TWISTED L² INVARIANTS OF NON-SIMPLY CONNECTED MANIFOLDS AND ASYMPTOTIC L² MORSE INEQUALITIES.

VARGHESE MATHAI AND MIKHAIL SHUBIN

Dedicated to Mark Iosifovich Vishik on the occasion of his 75th birthday

ABSTRACT. We develop the theory of *twisted* L^2-cohomology and twisted spectral invariants for flat Hilbertian bundles over compact manifolds. They can be viewed as functions on $H^1(M, \mathbb{R})$ and they generalize the standard notions. A new feature of the twisted L^2-cohomology theory is that in addition to satisfying the standard L^2 Morse inequalities, they also satisfy certain *asymptotic* L^2 Morse inequalities. These reduce to the standard Morse inequalities in the finite dimensional case, and when the Morse 1-form is exact. We define the *extended* twisted L^2 de Rham cohomology and prove the *asymptotic* L^2 Morse-Farber inequalities, which give quantitative lower bounds for the Morse numbers of a Morse 1-form on M.

INTRODUCTION

Let M be a compact closed C^∞-manifold. A C^∞-function $f : M \to \mathbb{R}$ is called *Morse function* if any critical point x of f (i.e. point $x \in M$ such that $df(x) = 0$) is non-degenerate. This means that the determinant of the Hessian does not vanish, i.e.

$$\det\left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right) \neq 0,$$

where the partial derivatives are taken in local coordinates. It follows that all critical points are isolated, therefore there is only finite number of them. The *index* of the critical point x is the number of the negative eigenvalues of the Hessian. Denote $m_j = m_j(f)$ the number of critical points with the index j.

M.Morse discovered a connection between the behavior of f and topology of M. In particular, the numbers m_j (which are called *Morse numbers*) are related with the (real) Betti numbers $b_j = b_j(M)$ by the *Morse inequalities*

$$(0.1) \qquad m_j(f) \geq b_j(M), \quad j = 1, \ldots, n,$$

where $n = \dim_{\mathbb{R}} M$. There are also more general inequalities

$$(0.2) \qquad \sum_{p=0}^{k}(-1)^{k-p}m_p \geq \sum_{p=0}^{k}(-1)^{k-p}b_p, \quad k = 1, \ldots n.$$

Note that adding two inequalities (0.2) with $k = j - 1$ and $k = j$, we obtain (0.1).

The Morse inequalities can be applied to estimate the Morse numbers m_j from below. For example, it follows from (0.1) that any Morse function of the 2-dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ should have at least 2 saddle points (critical points of index 1). In other direction, knowing explicitly a Morse function on M, we can estimate its Betti numbers from above. This has important applications e.g. in complex analysis.

We refer to [Mi] for an excellent exposition of the classical Morse theory and its applications.

The Morse theory has numerous generalizations, developments and applications. We will only discuss the directions which are most relevant for this paper.

E.Witten [Wi] suggested a new proof of the Morse inequalities, which is completely analytic. He suggested to deform the de Rham complex by replacing the de Rham external differential by a deformed differential

$$(0.3) \qquad d_s = \exp(-sf) d \exp(sf) = d + se(df), \quad s \gg 0,$$

where f is a Morse function on M, $e(df)$ is the operator of the external multiplication of forms by the 1-form df. Though the dimensions of the cohomology spaces do not change under this deformation, the Laplacians acquire a big parameter s and it becomes possible to apply semiclassical asymptotics with $h = 1/s$. The explicit form of the deformed Laplacians shows that their eigenfunctions with small eigenvalues become localized near the critical points of f. The number of small eigenvalues (multiplicities taken into account) in forms of degree j can be calculated and it equals m_j. Since 0 is among these eigenvalues, this implies the inequalities (0.1), because the de Rham cohomology space is isomorphic to the space of harmonic forms of the corresponding degree due to the Hodge theory.

S.Novikov [Nov, Nov2] suggested to replace the Morse function f by a closed Morse 1-form on M. Such a form can be considered as "multivalued Morse function" on M. This theory was further developed by M.Farber [F2] and A.Pazhitnov ([P, P2]).

It is quite natural to consider more general deformations than (0.3). In particular, we can take the deformation

$$(0.4) \qquad d_s = d + se(\omega), \quad s \gg 0,$$

where ω is a 1-form which is not necessarily exact. If ω is closed, we arrive to a reinterpretation of the Novikov theory (with "multivalued Morse functions"), which was used by M.Farber and A.Pazhitnov. However it is also interesting to take ω which is not closed, in spite of the fact that then $d_s^2 \neq 0$. The case when ω is dual to a Killing vector field, was considered by Witten [Wi]. A more general situation which leads to Morse-type inequalities for arbitrary vector fields was studied by Novikov (see Appendix to [NS] and also [Sh3]).

It was noticed by S.Novikov and M.Shubin [NS] that the Betti numbers b_j in (0.1) and (0.2) can be replaced by so-called L^2 Betti numbers $\bar{b}_j$ (or von Neumann Betti numbers) which were introduced by M.Atiyah [A]. They can be defined as von Neumann dimensions (associated with the fundamental group $\pi_1(M)$) of the spaces of L^2 harmonic forms on the universal covering $\widetilde{M}$ of M, where $\widetilde{M}$ is considered with a Riemannian metric which is lifted from M (or, equivalently, a Riemannian metric which is invariant under the action of $\pi_1(M)$ on $\widetilde{M}$ defined by the deck transformations). J.Dodziuk [Do] proved that the L^2 Betti numbers are homotopy invariants of M.

The L^2 Morse inequalities can be applied e.g. to prove that some topological require-
ments, imposed on M, imply existence of non-trivial L^2 harmonic forms on $\widetilde{M}$ ([NS]).

The Witten method can be applied to the L^2 situation as well ([Sh]) in spite of the fact
that the universal covering is generally non-compact and the Laplacians have continuous
spectrum. The main tool is a variational priciple (see e.g. [ES]) which works for von
Neumann dimensions.

In L^2 situation Morse theory is also naturally connected with the spectrum-near-zero
phenomenon in topology which was discovered by S.Novikov and M.Shubin (see [NS2]
and also [ES, E, GS, Lo, LL, F]). M.Gromov observed that if 0 is in the spectrum of
the Laplacian on p-forms on $\widetilde{M}$, then $m_p(f) > 0$ for any Morse function $f : M \to \mathbb{R}$, in
spite of the fact that it might happen that in this situation $b_p = \bar{b}_p = 0$, so the positivity
of m_p does not follow neither from the classical Morse inequalities, nor from their L^2-
version. A quantitative version of this observation was given by M.Farber [F] with the
help of his extended cohomology theory which puts the spectrum-near-zero invariants into
a cohomological context.

A more general context for L^2-cohomology is a flat vector bundle $\mathcal{E}$ on a compact
manifold M, such that the fiber E is a Hilbert module over a finite von Neumann algebra
$\mathcal{A}$. Such a bundle is called *Hilbertian bundle*. The L^2-functions and L^2-forms on the
universal covering of M can be interpreted as sections of such a bundle with the fiber
$\ell^2(\pi)$, where $\pi = \pi_1(M)$ is considered as a discrete group, $\ell^2(\pi)$ is the Hilbert space of
square-summable complex-valued functions on π.

In this paper we study the relationship between the Morse theory of closed 1-forms on
M and a *twisted* L^2-cohomology of a flat Hilbertian bundle over M. Here "twisted" means
that the covariant derivative of the flat connection is deformed by adding $e(\omega)$, where ω
is a closed form on M.

A new feature of the twisted L^2-cohomology is that in addition to the standard L^2
Morse inequalities we also have *asymptotic L^2 Morse inequalities*. Here the big parameter
is provided by the Witten-type deformation of the type (0.4) (with a closed 1-form ω) for
the covariant derivative. We also study the twisted analogue of the spectrum-near-zero
invariants. All these invariants can be viewed as functions on $H^1(M, \mathbb{R})$ possessing some
upper semi-continuity properties.

We begin with a review of some background material on Hilbertian modules over a von
Neumann algebra and flat Hilbertian bundles over a compact manifold M in section 1.
In section 2 we introduce a *twisted* analogue of all L^2-invariants associated with a flat
Hilbertian bundle $\mathcal{E} \to M$ over M, and briefly discuss their main properties. In section
3 we review the Morse theory of closed 1-forms on M and prove the standard-type L^2
Morse inequalities (see [NS] and [Sh] for the case of the regular representation), as well as
the *asymptotic L^2 Morse inequalities*, and we also give some applications. The proofs are
based on an analogue of the Witten deformation technique, which has also been applied
in the L^2-context by Burghelea, Friedlander, Kappeler, McDonald [BFKM] and Shubin
[Sh], in situations which are somewhat different to those considered here. In section 4 we
briefly review virtual Hilbertian modules as objects in the extended category introduced
by Farber [F]. Then, acting similarly to [Sh2], we define the extended twisted L^2 de Rham
cohomology. The main result here is the *asymptotic L^2 Morse-Farber inequalities*, which

give quantitative lower bounds for the Morse numbers of a Morse 1-form on M. In section 5 we end with some calculations and further applications.

On completing a preliminary version of our paper, we received a preprint of M. Braverman and M. Farber [BF], which also proved the asymptotic standard L^2 Morse inequalities, but only for the special case of *residually finite* fundamental groups, and using in an essential way the results of Lück [Lu]. In this case, they also prove the degenerate asymptotic L^2 Morse inequalities.

1. Preliminaries

In this section we establish the main notation of the paper, and review some basic facts about Hilbertian $\mathcal{A}$-modules. (See [F, CFM] for details.) We refer to [Di, T] for the necessary definitions on von Neumann algebras.

Let $\mathcal{A}$ be a finite von Neumann algebra with a fixed finite, normal and faithful trace $\tau : \mathcal{A} \to \mathbb{C}$. We will always assume that this trace is normalized i.e. $\tau(1) = 1$. The involution in $\mathcal{A}$ will be denoted $*$.

By $\ell^2(\mathcal{A})$ we denote the completion of $\mathcal{A}$ with respect to the scalar product $\langle a, b \rangle = \tau(b^*a)$, for $a, b \in \mathcal{A}$. For example, let Γ be a finitely generated discrete group and $\ell^2(\Gamma)$ denote the Hilbert space of square summable functions on Γ. Let ρ denote the left regular representation of Γ on $\ell^2(\Gamma)$. This extends linearly to a representation of the group algebra $\mathbb{C}(\Gamma)$. The weak closure of $\rho(\mathbb{C}(\Gamma))$ is called the group von Neumann algebra, denoted by $\mathcal{U}(\Gamma)$. The trace τ is given by evaluation at the identity element of Γ, i.e.

$$\tau(a) = \langle a\delta_e, \delta_e \rangle, \quad a \in \mathcal{U}(\Gamma),$$

where e is the neutral element in Γ, $\delta_e \in \ell^2(\Gamma)$, $\delta_e(x) = 1$ if $x = e$, and 0 otherwise.

Recall that a *Hilbert module* over $\mathcal{A}$ is a Hilbert space M together with a continuous left $\mathcal{A}$-module structure such that there exists an isometric $\mathcal{A}$-linear embedding of M into $\ell^2(\mathcal{A}) \bar\otimes H$, for some Hilbert space H. Note that this embedding is not part of the structure. A Hilbert module M is *finitely generated* if it admits an embedding $M \to \ell^2(\mathcal{A}) \otimes H$ as above with finite-dimensional H. Note that a Hilbert module comes with a particular scalar product.

A *Hilbertian module* is a topological vector space M with continuous left $\mathcal{A}$-action such that there exists a scalar product $\langle \ , \ \rangle$ on M which generates the topology of M and such that M together with $\langle \ , \ \rangle$ and with the $\mathcal{A}$-action is a Hilbert module.

If M is a Hilbertian module, then any scalar product $\langle \ , \ \rangle$ on M with the above properties will be called *admissible*. It can be proved that any other choice of an admissible scalar product gives an isomorphic Hilbert module ([CFM]) but such choice introduces an additional structure. The situation here is similar to the case of finite-dimensional vector spaces: any choice of a scalar product on a vector space produces an isomorphic Euclidean vector space.

Let M be a Hilbertian module and let $\langle \ , \ \rangle$ be an admissible scalar product. Then $\langle \ , \ \rangle$ must be compatible with the topology on M and with the $\mathcal{A}$-action. The last condition means that the involution on $\mathcal{A}$ determined by the scalar product $\langle \ , \ \rangle$ coincides with the involution of the von Neumann algebra $\mathcal{A}$:

$$\langle \lambda \cdot v, w \rangle = \langle v, \lambda^* \cdot w \rangle$$

for any $v, w \in M$, $\lambda \in \mathcal{A}$.

If $\langle \ , \ \rangle$ and $\langle \ , \ \rangle_1$ are two admissible scalar products on a Hilbertian module M then the Hilbert modules $(M, \langle \ , \ \rangle)$ and $(M, \langle \ , \ \rangle_1)$ are isomorphic. Therefore we can define *finitely generated Hilbertian modules* as those for which the corresponding Hilbert modules (obtained by a choice of an admissible scalar product) are finitely generated. Note that the *von Neumann dimension* of a Hilbertian module M, denoted $\dim_\tau(M)$, is also well defined (we will recall the definition later).

A *morphism* of Hilbertian modules is a continuous linear map $f : M \to N$, commuting with the $\mathcal{A}$-action. Note that the kernel of any morphism f is again a Hilbertian module. Also, the closure of the image $\mathrm{cl}(\mathrm{Im}\,(f))$ is a Hilbertian module.

Let $\mathcal{B} = B_\mathcal{A}(M)$ denote the set of endomorphisms of M as Hilbertian module. It can also be described as the commutant of the action of $\mathcal{A}$ on M, so we will sometimes refer to it simply as the *commutant*.

Any choice of an admissible scalar product $\langle \ , \ \rangle$ on M defines obviously a $*$-operator on $\mathcal{B}$ (by assigning to an operator its adjoint) and turns $\mathcal{B}$ into a von Neumann algebra. Note that this involution $*$ depends on the scalar product $\langle \ , \ \rangle$ on M; if we choose another admissible scalar product $\langle \ , \ \rangle_1$ on M then the new involution will be given by

$$f \mapsto A^{-1} f^* A \qquad \text{for} \quad f \in \mathcal{B},$$

where $A \in \mathcal{B}$ is the operator defined by $\langle v, w \rangle_1 = \langle Av, w \rangle$ for $v, w \in M$.

If M is finitely generated, then the trace $\tau : \mathcal{A} \to \mathbb{C}$ determines canonically a trace on the commutant

$$\mathrm{Tr}_\tau : \mathcal{B} = \mathcal{B}_\mathcal{A}(M) \to \mathbb{C}$$

which is finite, normal, and faithful.

We now briefly describe this trace. Suppose first that M is *free*, that is, M is isomorphic to $l^2(\mathcal{A}) \otimes \mathbb{C}^k$ for some k. Then the commutant $\mathcal{B}$ can be identified with the set of all $k \times k$-matrices over the algebra $\mathcal{A}$, acting from the right on $l^2(\mathcal{A}) \otimes \mathbb{C}^k$ (the last module is viewed as the set of row-vectors with components in $l^2(\mathcal{A})$). If $\alpha \in \mathcal{B}$ is an element represented by a $k \times k$ matrix (α_{ij}), then one defines

$$\mathrm{Tr}_\tau(\alpha) = \sum_{i=1}^{k} \tau(\alpha_{ii})$$

This formula gives a trace on $\mathcal{B}$ which is finite, normal and faithful. Note also that this trace Tr_τ does not depend on the representation of M as the product $l^2(\mathcal{A}) \otimes \mathbb{C}^k$.

Suppose now that M is an arbitrary finitely generated Hilbertian module. Then M can be embedded into a free module F as a direct summand. Let $\pi_M : F \to M$ and $i_M : M \to F$ denote the projection and embedding correspondingly. Then for $f \in \mathcal{B}_\mathcal{A}(M)$ the formula

$$\mathrm{Tr}_\tau(f) = \mathrm{Tr}_\tau(i_M \circ f \circ \pi_M)$$

defines a trace

$$\mathrm{Tr}_\tau : \mathcal{B}_\mathcal{A}(M) \to \mathbb{C},$$

satisfying

$$\mathrm{Tr}_\tau(fg) = \mathrm{Tr}_\tau(gf)$$

for all $f, g \in \mathcal{B}_{\mathcal{A}}(M)$. This trace is independent of all choices. In particular we can define $\mathcal{A}$-dimension of M by the formula

$$\dim_\tau M = \mathrm{Tr}_\tau(\mathrm{Id}_M).$$

1.1. **Hilbertian $\mathcal{A}$-complexes.** Let us consider a sequence

(1.1) $$L_\bullet : \ 0 \longrightarrow L_0 \xrightarrow{d_0} L_1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} L_n \longrightarrow 0,$$

where all L_j, $j = 1, \dots, n$, are Hilbertian $\mathcal{A}$-modules, d_j are closed densely defined linear operators which commute with the action of $\mathcal{A}$ in the sense that $d_j a = a d_j$ on $\mathrm{dom}(d_j)$ for all $a \in \mathcal{A}$. (Here $\mathrm{dom}(d_j)$ denotes the domain of d_j and it is a dense linear subspace in L_j.) In particular this means that $a(\mathrm{dom}(d_j)) \subset \mathrm{dom}(d_j)$ i.e. the domains $\mathrm{dom}(d_j)$ are $\mathcal{A}$-invariant.

Such a sequence is called a *complex of Hilbertian $\mathcal{A}$-modules* if $d_j d_{j-1} = 0$ on $\mathrm{dom}(d_{j-1})$ for all j. In particular this means that $\mathrm{Im}\, d_{j-1} \subset \mathrm{Ker}\, d_j$. Note that $\mathrm{Ker}\, d_j$ is always closed.

We will call a complex (1.1) *finite* if all Hilbertian modules L_j are finitely generated and all differentials d_j are (bounded) morphisms of Hilbertian $\mathcal{A}$-modules.

The reduced L^2-cohomology groups of $L_\bullet$ are defined as the Hilbertian $\mathcal{A}$-modules

$$H_{(2)}^p(L) = \frac{\mathrm{Ker}\, d_p}{\mathrm{cl}(\mathrm{Im}\, d_{p-1})}, \quad p = 0, 1, \dots, n.$$

(Here by definition d_{-1} and d_n are zero morphisms.) Denote $m_p = \dim_\tau L_p$, $\bar{b}_p = \dim_\tau H_{(2)}^p(L)$.

The following Lemmas are well known:

Lemma 1.1. *Suppose that $m_j < \infty$ for all j. Then*

$$\sum_{j=0}^n (-1)^j m_j = \sum_{j=0}^n (-1)^j \bar{b}_j.$$

Lemma 1.2. *If $m_j < \infty$ for all j then*

$$\sum_{j=0}^p (-1)^{p-j} m_j \geq \sum_{j=0}^p (-1)^{p-j} \bar{b}_j$$

for every $p = 0, \dots n$.

The proofs do not differ from the proofs of the corresponding statements for the von Neumann algebra $\mathcal{A} = \mathbb{C}$ (and for the spaces L_j which are finite-dimensional in the usual sense) except almost isomorphisms should be used instead of usual isomorphisms (see [Sh] for more details).

If $M_\bullet$ and $N_\bullet$ are Hilbertian $\mathcal{A}$-complexes, then a *morphism of Hilbertian $\mathcal{A}$-complexes* $f : M_\bullet \to N_\bullet$ is a sequence $f_k : M_k \to N_k$ of morphisms of Hilbertian $\mathcal{A}$-modules such that $f_{k+1} d_k w = d_k f_k w$ for all $w \in \mathrm{dom}(d_k)$. A *homotopy* between two morphisms $f, g : M_\bullet \to N_\bullet$ is a sequence of morphisms of Hilbertian $\mathcal{A}$-modules $T_k : M_k \to N_{k-1}$ such that $f_k - g_k = T_{k+1} d_k + d_{k-1} T_k$ on $\mathrm{dom}(d_k)$. Homotopy is an equivalence relation.

Two Hilbertian $\mathcal{A}$-complexes $M_\bullet$ and $N_\bullet$ are said to be homotopy equivalent if there exist morphisms $f : M_\bullet \to N_\bullet$ and $g : N_\bullet \to M_\bullet$ such that fg and gf are homotopic to the identity morphisms of $N_\bullet$ and $M_\bullet$ respectively. Homotopy equivalence is an equivalence relation.

If $M_\bullet$ is a Hilbertian $\mathcal{A}$-complex, then we can define functions $F_k(\lambda, M)$ as

$$F_k(\lambda, M) \equiv \sup \left(\dim_\tau L : L \in \mathcal{S}_\lambda^{(k)}(M) \right)$$

where $\mathcal{S}_\lambda^{(k)}(M)$ denotes the set of all closed $\mathcal{A}$-invariant subspaces of $M_k / \ker d_k$ such that $L \subset \mathrm{dom}(d_k) / \ker d_k$ and $\|d_k w\| \leq \sqrt{\lambda} \|w\|$ for $w \in L$. (The norm on the right hand side is the quotient norm). Then $\lambda \mapsto F_k(\lambda, M)$ is an increasing function on $\mathbb{R}$ with values in $[0, \infty]$ and $F_k(\lambda, M) = 0$ if $\lambda < 0$.

Given a Hilbertian $\mathcal{A}$-complex $M_\bullet$, consider the *Laplacian* $\Delta_k \equiv d_{k-1}\delta_{k-1} + \delta_k d_k$, where δ_k denotes the L^2 adjoint of d_k. It is a self-adjoint operator in M_k and it has the spectral decomposition

$$\Delta_k = \int_0^\infty \lambda dE_\lambda.$$

Then the von Neumann spectral density function is defined as

$$N_k(\lambda, M) = \mathrm{Tr}_\tau E_\lambda,$$

and can be expressed through the functions F_k as follows ([GS])

$$N_k(\lambda, M) = F_{k-1}(\lambda, M) + F_k(\lambda, M) + b_{(2)}^k(M),$$

where $b_{(2)}^k(M) = \dim_\tau \mathrm{Ker}\, \Delta_k$ are the L^2 Betti numbers.

Two functions $F(\lambda)$ and $G(\lambda)$ on $(0, \infty)$ satisfy $F \prec\!\!\prec G$ if there exist positive constants C, λ_0 such that $F(\lambda) \leq G(C\lambda)$ for all $\lambda \in (0, \lambda_0)$.

If $F \prec\!\!\prec G$ and $G \prec\!\!\prec F$, then F and G are said to be dilationally equivalent and we write $F \sim G$. Roughly speaking, in this case the small λ asymptotics of $F(\lambda)$ and $G(\lambda)$ are the same. The following basic abstract theorem is due to Gromov and Shubin [GS]:

Theorem 1.3. *Let $f : M \to N$ and $g : N \to M$ be morphisms of Hilbertian $\mathcal{A}$-complexes such that gf is homotopic to the identity morphism of M. Then $F_k(\lambda, M) \prec\!\!\prec F_k(\lambda, N)$ and $b_{(2)}^k(M) \leq b_{(2)}^k(N)$ for all k. Hence if M and N are homotopy equivalent, then $F_k(\lambda, M) \sim F_k(\lambda, N)$ and $b_{(2)}^k(M) = b_{(2)}^k(N)$.*

1.2. **Flat Hilbertian $\mathcal{A}$-Bundles.** Let E be a finitely generated Hilbertian $(\mathcal{A} - \pi)$-bimodule. This means first that $\mathcal{A}$ acts on E from the left, so that with respect to this action E is a finitely generated Hilbertian module; and second that π is a discrete group acting on E from the right and the action of π commutes with that of $\mathcal{A}$. A Hilbertian $(\mathcal{A} - \pi)$-bimodule E is said to be *unitary* if there exists an admissible scalar product $\langle\ ,\ \rangle$ on E such that the action of π on E preserves this scalar product.

Let M be a connected, closed, smooth manifold with fundamental group π. Let $\widetilde{M}$ denote the universal covering of M. A *flat Hilbertian $\mathcal{A}$-bundle with fiber E over M* is an associated bundle $p : \mathcal{E} \to M$. This means that $\mathcal{E} = (E \times \widetilde{M})/\sim$ with its natural projection onto M, where $(v, x) \sim (vg^{-1}, gx)$ for all $g \in \pi$, $x \in \widetilde{M}$ and $v \in E$. Then

$p : \mathcal{E} \to M$ is a locally trivial bundle of topological vector spaces (cf. chapter 3 of [La]) which has a natural fiberwise left action of $\mathcal{A}$.

A flat Hilbertian $\mathcal{A}$-bundle $\mathcal{E} \to M$ over M, with fiber E, is said to be *unitary* if the Hilbertian $\mathcal{A}$ -module E is unitary.

Any smooth section s of $\mathcal{E} \to M$ can be uniquely represented by a smooth equivariant map $\phi : \widetilde{M} \to E$, where "equivariant" means that $\phi(gx) = \phi(x)g^{-1}$ for all $g \in \pi$ and $x \in \widetilde{M}$. Given such a map ϕ, the corresponding section assigns to every $y \in M$, the equivalence class of the pair $(x, \phi(x))$, where x is a lifting of the point y.

Given a flat Hilbertian $\mathcal{A}$-bundle $\mathcal{E} \to M$ over a closed connected manifold M, one can consider the space of smooth differential j-forms on M with values in $\mathcal{E}$; this space will be denoted by $\Omega^j(M, \mathcal{E})$. It is naturally defined as a left $\mathcal{A}$-module and can be written as

$$(1.2) \qquad \Omega^j(M, \mathcal{E}) = C^\infty(M, \mathcal{E}) \bigotimes_{C^\infty(M)} \Omega^j(M) \,,$$

where $C^\infty(M, \mathcal{E})$ is the set of all C^∞-sections of $\mathcal{E}$ over M. An element of $\Omega^j(M, \mathcal{E})$ can be also uniquely represented as a π-invariant element in

$$(1.3) \qquad C^\infty(\widetilde{M}, E) \bigotimes_{C^\infty(\tilde{M})} \Omega^j(\widetilde{M})$$

with respect to the total (diagonal) action of π. Here $C^\infty(\widetilde{M}, E)$ is the space of E-valued C^∞-functions on $\widetilde{M}$, and $\Omega^j(\widetilde{M})$ is the space of C^∞-forms of degree j on $\widetilde{M}$.

An $\mathcal{A}$-*linear connection* on a flat Hilbertian $\mathcal{A}$-bundle $\mathcal{E}$ is defined as an $\mathcal{A}$-homomorphism

$$\nabla : \Omega^j(M, \mathcal{E}) \to \Omega^{j+1}(M, \mathcal{E})$$

which is given for all j and satisfies the Leibniz rule

$$\nabla(f\omega) = df \wedge \omega + f\nabla(\omega)$$

for any $\mathcal{A}$-valued function f on M and for any $\omega \in \Omega^j(M, \mathcal{E})$. This connection is called *flat* if $\nabla^2 = 0$. On a flat Hilbertian $\mathcal{A}$-bundle $\mathcal{E}$, as defined above, there is a *canonical flat $\mathcal{A}$-linear connection* ∇ which is given as follows: under the identification of $\Omega^j(M, \mathcal{E})$ given in the previous paragraph, one defines the connection ∇ to be the de Rham exterior derivative acting on the second factor in (1.3).

1.3. Hermitian metrics and L^2 scalar products.

A Hermitian metric on a flat Hilbertian $\mathcal{A}$-bundle $p : \mathcal{E} \to M$ is a smooth family of admissible scalar products on the fibers. Any Hermitian metric on $p : \mathcal{E} \to M$ defines a wedge-type product

$$\wedge : \Omega^p(M, \mathcal{E}) \otimes \Omega^q(M, \mathcal{E}) \to \Omega^{p+q}(M)$$

similar to the finite dimensional case (see e.g. (1.20), Chapter 3, in [We]). Note that this product is antilinear with respect to the second factor, and it is Hermitian-antisymmetric i.e.

$$\alpha \wedge \beta = (-1)^{pq}\overline{\beta \wedge \alpha}, \quad \alpha \in \Omega^p(M, \mathcal{E}), \ \beta \in \Omega^q(M, \mathcal{E}) \,.$$

Suppose we are also given a Riemannian metric on M. Then we can define the Hodge star-operator

$$* : \Omega^j(M, \mathcal{E}) \to \Omega^{n-j}(M, \mathcal{E} \otimes o(M)),$$

where $o(M)$ is the orientation bundle of M which is a real line-bundle (trivial if M is oriented). The operator $*$ is a complex linear operator defined as the complexification of the standard real Hodge star operator. It acts on form-coefficients (without affecting the fiber) i.e.

$$*(f \otimes \omega) = f \otimes (*\omega), \quad f \in C^\infty(M, \mathcal{E}), \ \omega \in \Omega^\bullet(M).$$

The Hermitian metric on $p : \mathcal{E} \to M$ together with a Riemannian metric on M determines a scalar product on $\Omega^i(M, \mathcal{E})$ in the standard way. Namely, one sets

$$(\omega, \omega') \ = \ \int_M \omega \wedge *\omega'$$

(cf. [We], Section 2 in Chapter 5). consisting

With this scalar product $\Omega^i(M, \mathcal{E})$ becomes a pre-Hilbert space. Define the space of L^2 differential j-forms on M with coefficients in $\mathcal{E}$, which is denoted by $\Omega^j_{(2)}(M, \mathcal{E})$, to be the Hilbert space completion of $\Omega^j(M, \mathcal{E})$. We will tend to ignore the scalar product on $\Omega^j_{(2)}(M, \mathcal{E})$ and view it as a Hilbertian $\mathcal{A}$ module (not necessarily finitely generated).

The connection ∇ on $\mathcal{E}$ extends by closure to a closed, unbounded, densely defined operator $\nabla : \Omega^j_{(2)}(M, \mathcal{E}) \to \Omega^{j+1}_{(2)}(M, \mathcal{E})$.

1.4. **Reduced L^2 cohomology.** Given a flat Hilbertian $\mathcal{A}$ bundle $p : \mathcal{E} \to M$, one defines the *reduced L^2 cohomology with coefficients in $\mathcal{E}$* as the quotient

$$H^j(M, \mathcal{E}) = \frac{\text{Ker } \nabla / \Omega^j_{(2)}(M, \mathcal{E})}{\text{cl}(\text{Im } \nabla / \Omega^{j-1}(M, \mathcal{E}))}.$$

Then $H^j(M, \mathcal{E})$ is naturally defined as a Hilbertian module over $\mathcal{A}$. The arguments given in [Do, Sh2] show that it coincides with the reduced combinatorial cohomology of M with coefficients in a locally constant sheaf, determined by $\mathcal{E}$.

1.5. **Hodge decomposition.** The Laplacian Δ_j acting on $\mathcal{E}$-valued L^2-forms of degree j on M is defined to be

$$\Delta_j = \nabla \nabla^* + \nabla^* \nabla : \Omega^j_{(2)}(M, \mathcal{E}) \to \Omega^j_{(2)}(M, \mathcal{E}),$$

where ∇^* denotes the Hilbert adjoint of ∇ with respect to the L^2 scalar product on $\Omega^\bullet_{(2)}(M, \mathcal{E})$. It is easy to see that the Laplacian is a self-adjoint operator. In fact it can be obtained as the closure from the operator given by the same expression on smooth forms.

Let $\mathcal{H}^j(M, \mathcal{E})$ denote the closed subspace of harmonic L^2-forms of degree j with coefficients in $\mathcal{E}$, that is, the kernel of Δ_j. Note that $\mathcal{H}^j(M, \mathcal{E})$ is a Hilbertian $\mathcal{A}$-module. By elliptic regularity (cf. section 2, [BFKM]), one sees that $\mathcal{H}^j(M, \mathcal{E}) \subset \Omega^j(M, \mathcal{E})$, that is, every L^2 harmonic j-form with coefficients in $\mathcal{E}$ is smooth. Standard arguments then show that one has the following Hodge decomposition (cf. [Do]; section 4, [BFKM] and also section 3, [GS])

$$\Omega^j_{(2)}(M, \mathcal{E}) = \mathcal{H}^j(M, \mathcal{E}) \oplus \text{cl}(\text{Im } \nabla / \Omega^{j-1}(M, \mathcal{E})) \oplus \text{cl}(\text{Im } \nabla^* / \Omega^{j+1}(M, \mathcal{E})).$$

Therefore it follows that the natural map

$$\mathcal{H}^j(M, \mathcal{E}) \to H^j(M, \mathcal{E})$$

is an isomorphism Hilbertian $\mathcal{A}$-modules. The corresponding L^2 Betti numbers are

$$b^j_{(2)}(M, \mathcal{E}) = \dim_\tau \left(H^j(M, \mathcal{E}) \right).$$

Definition 1.4. Let $\Delta_j = \int_0^\infty \lambda dE_j(\lambda)$ denote the spectral decomposition of the Laplacian. The *spectral density function* is defined to be $N_j(\lambda) = \text{Tr}_\tau(E_j(\lambda))$ and the *theta function* is defined to be $\Theta_j(t) = \int_{0+}^\infty e^{-t\lambda} dN_j(\lambda) = \text{Tr}_\tau(e^{-t\Delta_j}) - b^j_{(2)}(M, \mathcal{E})$.

These quantities are well defined because the projection $E_j(\lambda)$ and the heat operator $e^{-t\Delta_j}$ have smooth Schwartz kernels which are smooth sections of a bundle over $M \times M$ with fiber the commutant of E, cf. [BFKM, Luk]. The symbol Tr_τ denotes application of the canonical trace on the commutant to the restriction of the kernels to the diagonal followed by integration over the manifold M. This is a trace.

The *Novikov-Shubin invariants* are defined as follows (cf. [ES, Lo, LL]):

$$\alpha_j(M, \mathcal{E}) = \sup\{\beta \in I\!R : \Theta_j(t) \text{ is } O(t^{-\beta}) \text{ as } t \to \infty\} \in [0, \infty]$$
$$\overline{\alpha}_j(M, \mathcal{E}) = \inf\{\beta \in I\!R : t^{-\beta} \text{ is } O(\Theta_j(t))) \text{ as } t \to \infty\} \in [0, \infty].$$

2. Twisted L^2 cohomology and twisted L^2 invariants.

In this section we introduce the *twisted L^2* invariants associated with a flat Hilbertian $\mathcal{A}$-bundle $p : \mathcal{E} \to M$ over M. There are twisted analogues of all the invariants which were discussed in the previous section. They can be viewed as functions on $H^1(M, \mathbb{R})$ and will be used later in the asymptotic L^2 Morse inequalities in the next section.

We will assume that M is a compact Riemannian manifold. Let θ be a closed 1-form on M. Consider the twisted complex

$$(\Omega^\bullet_{(2)}(M, \mathcal{E}), \nabla_\theta)$$

where $\Omega^\bullet_{(2)}(M, \mathcal{E})$ denotes the space of L^2 differential forms on M with coefficients in $\mathcal{E}$, and the differential is given by

$$\nabla_\theta = \nabla + e(\theta)$$

where $e(\theta)$ denotes exterior multiplication by the 1-form θ. Clearly $\nabla^2_\theta = 0$. The closed 1-form defines a (real) representation of the fundamental group,

$$\rho_\theta : \pi_1(M) \to (0, \infty)$$
$$\rho_\theta(\alpha) = \exp\left(-\int_\alpha \theta\right).$$

This in turn defines a new flat Hilbertian $\mathcal{A}$-bundle $p : \mathcal{E}_\theta \to M$ over M, which can be described in several equivalent ways. One way is to note that the representation ρ_θ defines a flat real line bundle $p' : L_\theta \to M$ over M, where $L_\theta = (\mathbb{R} \times \widetilde{M})/\sim$ with its natural projection onto M. Here $(v, x) \sim (v\rho_\theta(g^{-1}), gx)$ for all $g \in \pi$, $x \in \widetilde{M}$ and $v \in \mathbb{R}$, $\widetilde{M}$ denotes the universal covering of M. Then the flat Hilbertian $\mathcal{A}$-bundle $\mathcal{E}_\theta$ over M, is defined to be the tensor product $\mathcal{E} \otimes L_\theta$.

The *twisted L^2 cohomology* is defined as

$$H^j_{(2)}(M, \mathcal{E}_\theta) = \frac{\text{Ker } \nabla_\theta|_{\Omega^j_{(2)}(M, \mathcal{E})}}{\text{cl}(\text{Im } \nabla_\theta|_{\Omega^{j-1}_{(2)}(M, \mathcal{E})})}.$$

Since $\mathcal{A}$ commutes with the differential ∇_θ, it follows that $H^j_{(2)}(M, \mathcal{E}_\theta)$ is a Hilbertian $\mathcal{A}$-module. Thus we can define the *twisted L^2 Betti numbers* as

$$b^j_{(2)}(M, \mathcal{E}_\theta) \equiv \dim_\tau(H^j_{(2)}(M, \mathcal{E}_\theta)).$$

We define the *twisted Laplacian*

$$\Delta_{\theta,j} = \nabla^*_\theta \nabla_\theta + \nabla_\theta \nabla^*_\theta$$

acting on $\Omega^j_{(2)}(M, \mathcal{E}_\theta)$. Here $\nabla^*_\theta = \nabla^* + i(V)$ denotes the formal adjoint of the operator ∇_θ and $i(V)$ denotes contraction with the vector field V which is the Riemannian dual to the 1-form θ. Then $\Delta_{\theta,j}$ is a formally self-adjoint operator with a unique self-adjoint extension, which we denote by the same symbol.

We denote by $\mathcal{H}^j(M, \mathcal{E}_\theta)$ the kernel of $\Delta_{\theta,j}$, and we refer to elements in $\mathcal{H}^j(M, \mathcal{E}_\theta)$ as *twisted L^2 harmonic j-forms*. ¿From the Hodge theorem discussed earlier we know that the Hilbertian $\mathcal{A}$-modules $\mathcal{H}^j(M, \mathcal{E}_\theta)$ and $H^j_{(2)}(M, \mathcal{E}_\theta)$ are isomorphic.

Let

$$\Delta_{\theta,j} = \int \lambda \, dE^j_\lambda(\theta)$$

denote the spectral decomposition of the twisted Laplacian and $e^\theta_j(\lambda, x, y)$ denote the Schwartz kernel of the projection $E^j_\lambda(\theta)$. Then by elliptic regularity theory, $e^\theta_j(\lambda, x, y)$ is smooth in x, y and we can define the twisted spectral distribution function

$$N_j(\lambda, \theta) = Tr_\tau(E^j_\lambda(\theta)) = \int_M Tr_\tau(e^\theta_j(\lambda, x, x)) dx$$

where the first Tr_τ denotes the von Neumann trace on the algebra of $\mathcal{A}$-invariant operators on $\Omega^*_{(2)}(M, \mathcal{E})$, and the trace under the integral sign means the trace in the corresponding fiber. It follows easily that

$$\lim_{\lambda \to 0^+} N_j(\lambda, \theta) = b^j_{(2)}(M, \mathcal{E}_\theta) \, .$$

Proposition 2.1. (a) *The dilatation class of $N_j(\lambda, \theta)$ depends only on the cohomology class $[\theta] \in H^1(M, \mathbb{R})$ of the form θ.*
   (b) *$b^j_{(2)}(M, \mathcal{E}_\theta)$ depends only on the cohomology class $[\theta] \in H^1(M, \mathbb{R})$ of the form θ.*

Proof. Let θ' be a closed 1-form on M which is cohomologous to θ i.e. $\theta' - \theta = dh$ where $h \in C^\infty(M)$. Then since

$$\nabla_{\theta'} = e^{-h} \nabla_\theta e^h \, ,$$

we see that the following diagram of complexes commutes:

$$
\begin{array}{ccccc}
\ldots \to & \Omega^j_{(2)}(M, \mathcal{E}_{\theta'}) & \xrightarrow{\nabla_{\theta'}} & \Omega^{j+1}_{(2)}(M, \mathcal{E}_{\theta'}) & \longrightarrow \ldots \\
 & \downarrow e^h & & \downarrow e^h & \\
\ldots \to & \Omega^j_{(2)}(M, \mathcal{E}_\theta) & \xrightarrow{\nabla_\theta} & \Omega^{j+1}_{(2)}(M, \mathcal{E}_\theta) & \to \ldots
\end{array}
$$

and e^h is a bounded $\mathcal{A}$-invariant map, that is, e^h is a morphism of Hilbert $\mathcal{A}$-modules. Thus by the abstract Theorem 1.3, $N_j(\lambda, \theta)$ and $N_j(\lambda, \theta')$ are dilatationally equivalent, completing the proof of part a).

The proof of part b) follows immediately from part a), by evaluating the spectral density functions at $s = 0$. $\qquad\square$

Proposition 2.2. *(Poincaré duality). Let $\mathcal{E} \to M$ be a flat unitary Hilbertian $\mathcal{A}$-bundle over M, and θ a closed 1-form on M. Then*

$$b^j_{(2)}(M, \mathcal{E}_\theta) = b^{n-j}_{(2)}(M, \mathcal{E}_{-\theta} \otimes o(M)),$$

where $n = \dim_\mathbb{R} M$, $o(M)$ is the orientation bundle of M.
*   In particular, if M is orientable then*

$$b^j_{(2)}(M, \mathcal{E}_\theta) = b^{n-j}_{(2)}(M, \mathcal{E}_{-\theta}).$$

Proof. Let $*$ denote the Hodge star operator, which induces a linear isomorphism

$$* : \Omega^j_{(2)}(M, \mathcal{E}_\theta) \to C^\infty(M, \mathcal{E}_{-\theta} \otimes \Lambda^{n-j} T^*(M) \otimes o(M)).$$

It is easy to see that $*$ intertwines the Laplacians on the bundles $\mathcal{E}_\theta \otimes \Lambda^j T^* M$ and $\mathcal{E}_{-\theta} \otimes \Lambda^{n-j} T^* M \otimes o(M)$. The result follows. $\qquad\square$

We next define the closed subspaces

$$E_j(M, \mathcal{E}_\theta) = \overline{\operatorname{Im} \nabla_\theta} \ \text{ in } \ \Omega^j_{(2)}(M, \mathcal{E}_\theta)$$

and

$$E^*_j(M, \mathcal{E}_\theta) = \overline{\operatorname{Im} \nabla^*_\theta} \ \text{ in } \ \Omega^j_{(2)}(M, \mathcal{E}_\theta) \, .$$

The Kodaira-Hodge decomposition theorem yields

$$\Omega^j_{(2)}(M, \mathcal{E}_\theta) = E_j(M, \mathcal{E}_\theta) \oplus \mathcal{H}^j(M, \mathcal{E}_\theta) \oplus E^*_j(M, \mathcal{E}_\theta).$$

This is an orthogonal decomposition with respect to the scalar product in $\Omega^j_{(2)}(M, \mathcal{E}_\theta)$. Since $\operatorname{Ker} \nabla_\theta$ is the orthogonal complement of $E^*_j(M, \mathcal{E}_\theta)$, we have

$$\operatorname{Ker} \nabla_\theta = E_j(M, \mathcal{E}_\theta) \oplus \mathcal{H}^j(M, \mathcal{E}_\theta) \, .$$

Since $\operatorname{Ker} \nabla^*_\theta$ is the orthogonal complement of $E_j(M, \mathcal{E}_\theta)$, we have

$$\operatorname{Ker} \nabla^*_\theta = \mathcal{H}^j(M, \mathcal{E}_\theta) \oplus E^*_j(M, \mathcal{E}_\theta) \, .$$

Let $F_j(\lambda, \theta)$ denote the spectral density function of the operator $\nabla^*_\theta \nabla_\theta$ acting on the subspace $E^*_j(M, \mathcal{E}_\theta)$ and $G_j(\lambda, \theta)$ denote the spectral density function of the operator $\nabla_\theta \nabla^*_\theta$ acting on the subspace $E_j(M, \mathcal{E}_\theta)$. Then clearly one has

$$N_j(\lambda, \theta) = G_j(\lambda, \theta) + b^j_{(2)}(M, \mathcal{E}_\theta) + F_j(\lambda, \theta) \, .$$

Lemma 2.3. $F_j(\lambda, \theta) = G_{j+1}(\lambda, \theta)$ *for $j = -1, 0, 1, \dots, n$, where $n = \dim_\mathbb{R} M$ and by fiat $F_{-1}(\lambda, \theta) = G_{n+1}(\lambda, \theta) = 0$.*

Proof. Note that

$$\nabla_{\theta,j} : E_j^*(M, \mathcal{E}_\theta) \longrightarrow E_{j+1}(M, \mathcal{E}_\theta)$$

is an (unbounded) almost isomorphism of Hilbert $\mathcal{A}$-modules, which intertwines the operators $\nabla_\theta^* \nabla_\theta$ acting on $E_j^*(M, \mathcal{E}_\theta)$ and $\nabla_\theta \nabla_\theta^*$ acting on $E_{j+1}(M, \mathcal{E}_\theta)$.

Let $U_{\theta,j}$ denote the unitary factor of $\nabla_{\theta,j}$ in its polar decomposition. Then

$$U_{\theta,j} : E_j^*(M, \mathcal{E}_\theta) \longrightarrow E_{j+1}(M, \mathcal{E}_\theta)$$

is a bounded isomorphism of Hilbert $\mathcal{A}$-modules which intertwines the same pair of operators as before. This proves the lemma. $\qquad\square$

We conclude that

$$N_j(\lambda, \theta) = F_{j-1}(\lambda, \theta) + b_{(2)}^j(M, \mathcal{E}_\theta) + F_j(\lambda, \theta).$$

Lemma 2.4. . *We can express*

$$F_j(\lambda, \theta) = \sup \Big\{ \ \dim_\tau L \mid L \text{ is a closed subspace of } \Omega_{(2)}^j(M, \mathcal{E}_\theta) / \operatorname{Ker} \nabla_\theta \text{ satisfying}$$

$$\|\nabla_\theta \omega\| \le \lambda \|\omega\|_q \quad \text{for all} \quad \omega \in L \Big\}.$$

Here the norm $\|.\|$ denotes the usual norm on $\Omega_2^{j+1}(M, \mathcal{E}_\theta)$, whereas $\|.\|_q$ denotes the norm in the quotient space $\Omega_{(2)}^j(M, \mathcal{E}_\theta) / \operatorname{Ker} \nabla_\theta$.

Proof. This is an immediate consequence of the isomorphism

$$\frac{\Omega_{(2)}^j(M, \mathcal{E}_\theta)}{\operatorname{Ker} \nabla_\theta} \cong E_j^*(M, \mathcal{E}_\theta)$$

and the variational principle of [ES]. $\qquad\square$

Corollary 2.5. . *The dilatation class of $F_j(\lambda, \theta)$ is independent of the choice of Riemannian metric on M.*

Proof. This is immediate from the expression for $F_j(\lambda, \theta)$ obtained in the previous lemma. $\qquad\square$

It is also clear that

$$b_{(2)}^j(M, \mathcal{E}_\theta) = \dim_\tau \left(H_{(2)}^j(M, \mathcal{E}_\theta) \right) = \dim_\tau \left(\frac{\operatorname{Ker} \nabla_\theta|_{\Omega_{(2)}^j(M, \mathcal{E}_\theta)}}{\operatorname{cl}(\operatorname{Im} \nabla_\theta|_{\Omega_{(2)}^{j-1}(M, \mathcal{E}_\theta)})} \right)$$

is independent of the choice of metric on M.

Corollary 2.6. . *The dilatation class of $N_j(\lambda, \theta)$ is independent of the choice of metric on M.*

We define the twisted Novikov-Shubin invariants for $j = 0, 1, \dots, \dim M$, as follows:

$$\alpha_j(M, \mathcal{E}_\theta) = \sup\{\beta \in \mathbb{R} : N_j(\lambda, \theta) - b_{(2)}^j(M, \mathcal{E}_\theta) \text{ is } O(\lambda^\beta) \text{ as } \lambda \to 0\}$$

$$\bar{\alpha}_j(M, \mathcal{E}_\theta) = \inf\{\beta \in \mathbb{R} : \lambda^\beta \text{ is } O(N_j(\lambda, \theta) - b_{(2)}^j(M, \mathcal{E}_\theta)) \text{ as } \lambda \to 0\}.$$

As an immediate consequence of Proposition 2.1 and Corollary 2.6, one has

Corollary 2.7. . $\alpha_j(M, \mathcal{E}_\theta)$ and $\bar{\alpha}_j(M, \mathcal{E}_\theta)$ are independent of the choice of Riemannian metric on M. Moreover, they depend only on the cohomology class $[\theta] \in H^1(M, \mathbb{R})$ of the form θ.

Define the *twisted von Neumann theta functions* as

$$\Theta_j(t, \theta) = \int_0^\infty e^{-t\lambda} dN_j(\lambda, \theta) = \text{Tr}_\tau(e^{-t\Delta_{\theta,j}}) - b^j_{(2)}(M, \mathcal{E}_\theta).$$

By the Tauberian theorem relating the large time t asymptotics of the von Neumann theta function $\Theta_j(t, \theta)$ to the small λ asymptotics of the twisted spectral density function $N_j(\lambda, \theta)$ (see Appendix in [GS]) one sees that

$$\alpha_j(M, \mathcal{E}_\theta) = \sup\{\beta \in \mathbb{R} : \Theta_j(t, \theta) \text{ is } O(t^{-\beta}) \text{ as } t \to \infty\},$$

$$\bar{\alpha}_j(M, \mathcal{E}_\theta) = \inf\{\beta \in \mathbb{R} : t^{-\beta} \text{ is } O(\Theta_j(t, \theta)) \text{ as } t \to \infty\}.$$

3. ASYMPTOTIC L^2 MORSE INEQUALITIES.

In this section, we prove Morse inequalities of two types for the twisted L^2 Betti numbers. Except of the standard type inequalities we also establish new asymptotic Morse inequalities. We refer the reader to Novikov [Nov2] and Pazhitnov [P] for discussions on the Morse theory of closed 1-forms as applied to finite-dimensional flat bundles over compact manifolds.

3.1. **Morse theory of closed 1-forms.** We now briefly review the Morse theory of closed 1-forms on M, which is one way to generalize the Morse theory of M by taking into account the fundamental group $\pi_1(M)$ of M. ¿From now on we assume that M be a smooth, closed, connected, orientable manifold and let $p : \widetilde{M} \to M$ be the universal cover of M.

Definition 3.1. A closed 1-form θ on M is said to be a *Morse 1-form* if $p^*(\theta) = df$ where f is a Morse function on $\widetilde{M}$.

Remarks 3.2. Another obviously equivalent way is to say that θ is a Morse 1-form if locally (i.e. in a neighborhood of every given point in M) the form θ can be presented as $\theta = df$ where f is a Morse function in this neighborhood. In fact it is sufficient to have such a presentation in a neighborhood of any zero of θ.

Assume for the rest of this section that θ is a Morse 1-form and that $p^*(\theta) = df$ where f is a Morse function on $\widetilde{M}$.

Definition 3.3. The *index* of a zero point x of θ is by fiat the index of a critical point $\widetilde{x}$ of f for any lift $\widetilde{x}$ of x.

The following local description for Morse 1-forms θ is just a reformulation of the classical Morse lemma (cf. [Mi]) and can be deduced from the equivalent definition of a Morse 1-form given in the remarks 3.2 above.

Lemma 3.4 (Morse lemma for Morse 1-forms)**.** *Let θ be a Morse 1-form on M.*

(a) *Suppose that p is a point such that $\theta_p \neq 0$. Then there is a chart $(\mathcal{U}, \phi)$ centered at p such that*

$$\left(\phi^{-1}\right)^* \theta \left(u_1, \dots, u_n\right) = du_1.$$

(b) *Suppose that p is a zero point of θ of index k. Then there is a chart $(\mathcal{U}, \phi)$ centered at p such that*

$$\left(\phi^{-1}\right)^* \theta \left(u_1, \dots, u_n\right) = -\sum_{j=1}^{k} u_j du_j + \sum_{j=k+1}^{n} u_j du_j.$$

Definition 3.5. . Let $m_k(\theta)$ denote the number of zero points of the Morse 1-form θ of index k. It is called the k-th *Morse number* of θ.

The following is the analogue of the classical result which states that on a compact closed manifold the set of Morse functions is open and dense in the space of all smooth functions.

Proposition 3.6 (Density of Morse 1-forms)**.** *Let M be a compact closed manifold. Then for every (de Rham) cohomology class $\alpha \in H^1(M; \mathbb{R})$ the set of Morse 1-forms $\theta \in \alpha$ is open and dense in the set of all the forms in this cohomology class.*

This proposition can be easily deduced from its classical analogue.

3.2. **Semiclassical asymptotics.** We will briefly describe an L^2-version of semiclassical asymptotics, which are similar to the ones which appear when we take the Witten deformation of the de Rham complex and consider the corresponding Laplacian (cf. [Wi, CFKS, HS]. For the case of the algebra $\mathcal{A}$ corresponding to the regular representation of $\pi_1(M)$, such asymptotics were first proved in [Sh]. The proofs given in [Sh] work in a more general situation which we need now.

Let $H = -hA + B + h^{-1}V$ be a second order differential operator acting in sections of a Hilbertian $\mathcal{A}$-bundle $\mathcal{F} \to M$ over M. Let F denote the Hilbertian $\mathcal{A}$-module which is the fiber of $\mathcal{F}$. We assume that all the coefficients in the local representations of H are smooth functions with values in $\mathrm{End}_{\mathcal{A}}(F)$, so H commutes with the fiberwise action of $\mathcal{A}$.

We will also assume that H satisfies requirements similar to the ones in [Sh]. Namely, we require that A is a second order elliptic operator with a negative principal symbol, B is a zeroth order operator, V is a non-negative potential function on M which has only nondegenerate zeroes and $h > 0$ is a small parameter.

Let $N_{\mathcal{A}}(\lambda; H)$ denote the von Neumann spectral density function of the operator H. Let K denote the *model operator* of H (cf. [Sh]), which is obtained as a direct sum of quadratic parts of H in all zeros of V. More precisely, let $\{\bar{x}_j, j = 1, \dots, N\}$ be the set of all zeros of V. Then $K = \sum_j K_j$, where

$$K_j : L^2(\mathbb{R}^n, F) \to L^2(\mathbb{R}^n, F)$$

corresponds to the zero $\bar{x}_j$, acts in the Hilbert space $L^2(\mathbb{R}^n, F)$ of all F-valued L^2-functions on $\mathbb{R}^n$ and has the form

$$K_j = -A_j^{(2)} + B_j + V_j^{(2)},$$

where all the components are obtained from H as follows. Let us fix local coordinates on M and a flat trivialization of $\mathcal{F}$ near $\bar{x}_j$. The second order term $A_j^{(2)}$ is a homogeneous

second order differential operator with constant coefficients (without lower order terms) obtained by isolating the second order terms in the operator A and freezing the coefficients of this operator at $\bar{x}_j$. The zeroth order term B_j is a constant $\mathcal{A}$-endomorphism of F which is obtained by freezing the coefficients of B at $\bar{x}_j$. The other zeroth order term $V_j^{(2)}$ is obtained by taking the quadratic part of V in the chosen coordinates near $\bar{x}_j$.

Let $\left\{ \mu_j : j = 1, 2, 3.... \right\}$ be the eigenvalues of the model operator K, $\mu_i \neq \mu_j$ for $i \neq j$, and r_j denote the multiplicity of μ_j. Let $\sigma(H)$ denote the L^2-spectrum of H, i.e. its spectrum in the Hilbert space of L^2-sections of $\mathcal{F}$. Then the following result is an easy generalization of the corresponding result in [Sh]:

Theorem 3.7 (Semiclassical Approximation)**.** *For any $R > 0$ and $\kappa \in (0, 1)$ there exist $C > 0$ and $h_0 > 0$ such that*

$$\sigma(H) \cap [-R, R] \subset \cup_{j=1}^{\infty}(\mu_j - Ch^{1/5}, \mu_j + Ch^{\kappa}) .$$

Moreover for any $j = 1, 2, 3, \ldots$ with $\mu_j \in [-R, R]$ and any $h \in (0, h_0)$ one has

$$N_{\mathcal{A}}(\mu_j + Ch^{\kappa}; H) - N_{\mathcal{A}}(\mu_j - Ch^{1/5}; H) = r_j .$$

Besides h^{κ} can be replaced by $\exp(-C^{-1}h^{-1+\varepsilon})$ and $h^{1/5}$ by h^{κ} if H is flat near all points $\bar{x}_j$ (i.e. if H coincides with K_j near $\bar{x}_j$ for every j).

This means that for small h the spectrum of H concentrates near the eigenvalues of the model operator K, and for every such eigenvalue, the von Neumann dimension of the spectral subspace of the operator H, corresponding to the part of the spectra near the eigenvalue, is exactly equal to the multiplicity of this eigenvalue.

We now state the main result of this section.

Theorem 3.8. *Let M be a compact manifold, f a Morse function on M and θ a closed 1-form on M. Let $\mathcal{E} \to M$ be a flat Hilbertian $\mathcal{A}$-bundle over M with fiber E which is a finitely generated Hilbertian $\mathcal{A}$-module.*

(1) *(Strong Morse Inequalities).*

$$(\dim_{\tau} E)^{-1} \sum_{j=0}^{k}(-1)^{k-j}b_{(2)}^{j}(M, \mathcal{E}_{\theta}) \leq \sum_{j=0}^{k}(-1)^{k-j}m_j(f)$$

   for $k = 0, 1, 2 \ldots, n$, with equality when $k = \dim_{\mathbb{R}} M = n$.

(2) *(Asymptotic Strong Morse Inequalities). Assume additionally that θ is a Morse form. Then for $s \gg 0$, one has*

$$(\dim_{\tau} E)^{-1} \sum_{j=0}^{k}(-1)^{k-j}b_{(2)}^{j}(M, \mathcal{E}_{s\theta}) \leq \sum_{j=0}^{k}(-1)^{k-j}m_j(\theta)$$

   for $k = 0, 1, 2 \ldots, n$, with equality when $k = n$.

Let α, β be closed 1-forms on M, so that α is a Morse form. Note that $\beta + s\alpha$ will be a closed Morse 1-form if s is sufficiently large. Consider the associated complex

$$\left(\Omega_{(2)}^{\bullet}(M, \mathcal{E}), \nabla_{\beta+s\alpha} \right)$$

The corresponding Laplacian $\Delta_{\beta+s\alpha,j} = \nabla^*_{\beta+s\alpha}\nabla_{\beta+s\alpha} + \nabla_{\beta+s\alpha}\nabla^*_{\beta+s\alpha}$ acts on $\Omega^j_{(2)}(M, \mathcal{E})$.

Define the Riemannian dual V of α which is a vector field on M satisfying $g(V, \cdot) = \alpha(\cdot)$. Let

$$\mathcal{L}_V : \Omega^\bullet(M, \mathcal{E}) \to \Omega^\bullet(M, \mathcal{E})$$

denote the Lie derivative acting in $\mathcal{E}$-valued forms on M, i.e.

$$\mathcal{L}_V(f \otimes \omega) = \nabla f \otimes \omega + f \otimes \mathcal{L}_V\omega, \quad f \in C^\infty(M, \mathcal{E}), \ \omega \in \Omega^\bullet(M) \ ,$$

where $\mathcal{L}_V\omega$ is defined as the usual Lie derivative, applied to a scalar form ω, ∇ is the flat connection on $\mathcal{E}$.

Lemma 3.9. $\Delta_{\beta+s\alpha,j} = \Delta_{\beta,j} + s\left(\mathcal{L}_V + \mathcal{L}_V^* + 2\langle\alpha,\beta\rangle\right) + s^2|\alpha|^2$ where $\mathcal{L}_V$ denotes the Lie derivative of the vector field V, defined as above, and $\mathcal{L}_V^*$ denotes its L^2 adjoint.

Proof. Since $(e(\alpha))^* = i(V)$, we obtain $\Delta_{\beta+s\alpha,j} = \left\{\nabla_\beta + se(\alpha), \nabla^*_\beta + si(V)\right\}_+$ where $\{A, B\}_+ = AB + BA$ denotes the anticommutator. Therefore

$$\Delta_{\beta+s\alpha,j} = \Delta_{\beta,j} + s\left\{\nabla_\beta, i(V)\right\}_+ + s\left\{e(\alpha), \nabla^*_\beta\right\}_+ + s^2\left\{e(\alpha), i(V)\right\}_+$$

$$= \Delta_{\beta,j} + s(\mathcal{L}_V + \mathcal{L}_V^* + 2\langle\alpha,\beta\rangle) + s^2|\alpha|^2 \ .$$

$\square$

Clearly

$$\frac{1}{s}\Delta_{\beta+s\alpha,j} = \frac{1}{s}\Delta_{\beta,j} + (\mathcal{L}_V + \mathcal{L}_V^* + 2\langle\alpha,\beta\rangle) + s|\alpha|^2$$

is of the form $H = -hA + B + h^{-1}V$ required to apply Theorem 3.7, where $h = \frac{1}{s}$, $A = -\Delta_{\beta,j}$ is independent of h, $B = \mathcal{L}_V + \mathcal{L}_V^* + 2\langle\alpha,\beta\rangle$ is a zeroth order operator and $V = |\alpha|^2$ is a non-negative function with non-degenerate zeros precisely at the zeros of α. Also all the terms commute with the action of $\mathcal{A}$. Denote by

$$K_{(j)} : \Omega^j_{(2)}(\mathbb{R}^n, E) \to \Omega^j_{(2)}(\mathbb{R}^n, E)$$

the corresponding model operator. (Here $\Omega^j_{(2)}(\mathbb{R}^n, E)$ is the Hilbert space of E-valued L^2-forms of degree j on $\mathbb{R}^n$.) For s large the operator $\frac{1}{s}\Delta_{\beta+s\alpha,j}$ develops gaps in its spectrum which persist for all large values of s. In the semiclassical limit, the spectrum of $\frac{1}{s}\Delta_{\beta+s\alpha,j}$ converges to the spectrum of a model operator $K_{(j)}$, which is the direct sum of harmonic oscillator type operators which operate near the zero points of α.

More precisely, if $E^j_\lambda(\beta + s\alpha)$ denotes the spectral projection of the operator $\frac{1}{s}\Delta_{\beta+s\alpha,j}$, then it follows from Theorem 3.7 that the von Neumann dimension of the spectral subspace $\mathrm{Im}\left(E^j_\varepsilon(\beta + s\alpha)\right)$ is a constant which equals $\dim_\tau(\mathrm{Ker}\ K_{(j)})$ for all $s > \left(\frac{C}{\varepsilon}\right)^5$. Note in particular that ε lies in the gap of the spectrum of $\frac{1}{s}\Delta_{\beta+s\alpha,j}$ when $s \gg 0$. The problem of calculating the eigenvalues (and their multiplicities) of the model operator $K_{(j)}$ actually reduces to the diagonalizing of B at the critical points of the Morse 1-form α. Note that this is a local calculation, which was performed by Witten (cf. [Wi, HS]) in the standard case. An obvious modification of Witten's arguments to the case of forms with coefficients in $\mathcal{E}$ leads to the following

Lemma 3.10. *The multiplicity of* 0 *as an eigenvalue of* $K_{(j)}$ *is*

$$\dim_\tau(\operatorname{Ker} K_{(j)}) = m_j(\alpha) \ \dim_\tau E.$$

Let us take $L^j = \operatorname{Im} E^j_\varepsilon(\beta + s\alpha)$ for $s \gg 0$. Then it can be proved as in [Sh2] that L^j is a finitely generated Hilbertian $\mathcal{A}$-module, and

$$\nabla_{\beta+s\alpha} : L^j \to L^{j+1}$$

is a bounded coboundary operator commuting with the $\mathcal{A}$-action; in particular, $\nabla^2_{\beta+s\alpha} = 0$. The Hilbertian $\mathcal{A}$-complex

$$L^\bullet : 0 \longrightarrow L^0 \xrightarrow{\nabla_{\beta+s\alpha}} L^1 \xrightarrow{\nabla_{\beta+s\alpha}} \ldots \xrightarrow{\nabla_{\beta+s\alpha}} L^n \longrightarrow 0$$

is bounded chain homotopy equivalent to $\left(\Omega^\bullet_{(2)}(M,\mathcal{E}), \nabla_{\beta+s\alpha}\right)$, and in particular, its reduced L^2-cohomology is $H^j_{(2)}(L^\bullet, \nabla_{\beta+s\alpha}) = H^j_{(2)}(M, \mathcal{E}_{\beta+s\alpha})$.

Now modifying an argument given in [Sh2] we obtain

Proposition 3.11. *Let* M *be a compact closed manifold,* α, β *closed 1-forms on* M*, and* α *is a Morse form. Let* $\mathcal{E} \to M$ *be a flat Hilbertian* $\mathcal{A}$-*bundle over* M*. Then the twisted de Rham complex of* L^2 *differential forms* $\left(\Omega^\bullet_{(2)}(M,\mathcal{E}), \nabla_{\beta+s\alpha}\right)$ *is bounded chain homotopy equivalent to the finitely generated Hilbertian* $\mathcal{A}$-*complex* $(L^\bullet, \nabla_{\beta+s\alpha})$.

Proof of Theorem 3.8 . By Lemmas 1.1 and 1.2, it follows that for $s \gg 0$ one has

$$(\dim_\tau E)^{-1} \sum_{j=0}^k (-1)^{k-j} b^j_{(2)}(M, \mathcal{E}_{\beta+s\alpha}) \le \sum_{j=0}^k (-1)^{k-j} m_j(\alpha)$$

with the equality for $k = n$.

Setting $\alpha = df$ and $\beta = \theta$, we observe that $[\beta + s\alpha] = [\theta]$ for all s, proving part (1).

Setting $\alpha = \theta$ and $\beta = 0$, we observe that $[\beta + s\alpha] = s[\alpha]$, proving part (2). $\qquad\square$

The equalities (the case $k = n$) in part (2) of Theorem 3.8 follow also from the stability of the (L^2)-index under deformations, so in fact it is not necessary to take $s \gg 0$. More precisely, one has

Proposition 3.12. *For any closed Morse 1-form* θ *on* M *one has*

$$(\dim_\tau E)^{-1} \sum_{j=0}^n (-1)^j b^j_{(2)}(M, \mathcal{E}_\theta) = \chi(M) = \sum_{j=0}^n (-1)^j m_j(\theta).$$

Proof. The first equality is in fact a particular case of the general L^2-index theorem by I.Singer [Si]. (See also [A] for the case of the algebra $\mathcal{A}$ generated by the regular representation of $\pi_1(M)$.) Firstly, observe that for the L^2-index of the operator $\nabla_\theta + \nabla^*_\theta$ we have

$$\begin{aligned}
\operatorname{index}_{L^2}(\nabla_\theta + \nabla^*_\theta) &= \dim_\tau \operatorname{Ker}(\nabla_\theta + \nabla^*_\theta)|_{\Omega^{even}_{(2)}(M,\mathcal{E})} - \dim_\tau \operatorname{Ker}(\nabla_\theta + \nabla^*_\theta)|_{\Omega^{odd}_{(2)}(M,\mathcal{E})} \\
&= \sum_{j=0}^n (-1)^j b^j_{(2)}(M, \mathcal{E}_\theta),
\end{aligned}$$

where $\Omega^{even}_{(2)}(M,\mathcal{E})$ denotes the even degree L^2 differential forms on M with values in $\mathcal{E}$, and $\Omega^{odd}_{(2)}(M,\mathcal{E})$ the odd degree ones. By the deformation invariance of the L^2-index, one

sees that $\text{index}_{L^2}(\nabla_\theta + \nabla_\theta^*) = \text{index}_{L^2}(\nabla + \nabla^*)$, where $\nabla + \nabla^*$ is considered as an operator in $\Omega_{(2)}^\bullet(M, \mathcal{E})$. Together with Theorem 3.8, one deduces the proposition. $\square$

The idea of the following proposition is due to M. Gromov. A quantitative version of this proposition will be given in the next section.

Proposition 3.13. *Let θ be a closed 1-form on M, and $\sigma(\Delta_{\theta,j})$ denote the L^2-spectrum of the twisted Laplacian $\Delta_{\theta,j}$.*

(a) *Suppose that $0 \in \sigma(\Delta_{\theta,j})$. Then $m_j(f) > 0$ for every Morse function f on M.*

(b) *Suppose that θ is a Morse form and $0 \in \sigma(\Delta_{s\theta,j})$ for $s \gg 0$. Then $m_j(\theta) > 0$.*

Proof. (a) The key idea is to use the abstract Theorem 1.3 (see also [GS]) which shows that 0-in-the-spectrum is a homotopy-invariant phenomenon. From this theorem we deduce first that the inclusions $0 \in \sigma(\Delta_{\theta+sdf,j})$, depending formally on s, are equivalent for different s because the corresponding complexes are isomorphic (see proof of Proposition 2.1). Secondly, for $s \gg 0$ the twisted complex $(\Omega_{(2)}^\bullet(M, \mathcal{E}), \nabla_{\theta+sdf})$ is homotopy equivalent to the complex $(L^\bullet, \nabla_{\theta+sdf})$ with $\dim_\tau L^j = m_j(f) \dim_\tau E$ due to Lemma 3.10 and Proposition 3.11. Since $m_j(f) = 0$ would imply that 0 is not in the spectrum of the Laplacian in the complex $(L^\bullet, \nabla_{\theta+sdf})$, the same is true in the homotopy equivalent complex $(\Omega_{(2)}^\bullet(M, \mathcal{E}), \nabla_{\theta+sdf})$, and (a) follows.

The proof of (b) can be done by similar arguments. $\square$

4. Asymptotic L^2 Morse-Farber inequalities

In this section we briefly review the extended category construction of Farber [F] and we use it to define the the extended twisted de Rham L^2-cohomology. (See also Lück [Lu2] for a different and purely algebraic approach to L^2-cohomology which is equivalent to the Farber's approach.) The main result of the section is the asymptotic L^2 Morse-Farber inequalities for the extended twisted de Rham L^2-cohomology. We shall assume in this section that $\mathcal{A}$ is a finite von Neumann algebra equipped with a finite, normal and faithful trace τ.

4.1. **The extended category.**
In [F] Farber used a P.Freyd construction to define an Abelian category $\mathcal{E}(\mathcal{A})$, which he called the extended category of Hilbertian $\mathcal{A}$ modules. It contains the usual additive category $\mathcal{H}(\mathcal{A})$ of Hilbertian $\mathcal{A}$ modules, which turns out to be the full subcategory of projective objects in $\mathcal{E}(\mathcal{A})$. We refer to [F] for further details on the following discussion about $\mathcal{E}(\mathcal{A})$.

An object in $\mathcal{E}(\mathcal{A})$ is a morphism $(\alpha : A' \to A)$ in $\mathcal{H}(\mathcal{A})$ (so A, A' are objects in $\mathcal{H}(\mathcal{A})$), and it is called a *virtual Hilbertian $\mathcal{A}$ module.*

A morphism

$$(\alpha : A' \to A) \longrightarrow (\beta : B' \to B)$$

in $\mathcal{E}(\mathcal{A})$ is defined by a morphism $f : A \to B$ in $\mathcal{H}(\mathcal{A})$ such that there exists a morphism $h : A' \to B'$ (in $\mathcal{H}(\mathcal{A})$) with $f\alpha = \beta h$. Two such morphisms $f, f' : A \to B$ define the same morphism in $\mathcal{E}(\mathcal{A})$ if and only if there exists a morphism $g : A \to B'$ such that $f - f' = \beta g$.

A Hilbertian $\mathcal{A}$ module A can be canonically viewed as a virtual Hilbertian $\mathcal{A}$ module $(0 \to A)$, and this correspondence is an embedding of $\mathcal{H}(\mathcal{A})$ into $\mathcal{E}(\mathcal{A})$.

A virtually Hilbertian $\mathcal{A}$-module $(\alpha : A' \to A)$ is said to be a *torsion* object in $\mathcal{E}(\mathcal{A})$ if $\overline{\operatorname{Im} \alpha} = A$. The Novikov-Shubin invariant is an invariant of a torsion object in $\mathcal{E}(\mathcal{A})$. A virtually Hilbertian $\mathcal{A}$ module $(\alpha : A' \to A)$ is a *projective* object in $\mathcal{E}(\mathcal{A})$ if and only if $\operatorname{Im} \alpha$ is closed.

The von Neumann dimension is an invariant of a projective object in $\mathcal{E}(\mathcal{A})$.

A virtually Hilbertian $\mathcal{A}$ module $(\alpha : A' \to A)$ has a canonically defined torsion part $(\alpha : A' \to \overline{\operatorname{Im} \alpha})$ and a canonically defined projective part $(0 \to A/\overline{\operatorname{Im} \alpha})$, and it is a direct sum of these two parts (though not canonically).

4.2. The extended L^2-cohomology. Let

$$C^\bullet : \quad \cdots \to C^{i-1} \to C^i \to C^{i+1} \to \dots$$

be a Hilbertian $\mathcal{A}$-complex (cf. section 1). It can be viewed as a complex of virtually Hilbertian $\mathcal{A}$-modules, i.e. as a complex in the extended category $\mathcal{E}(\mathcal{A})$. Then the i-th cohomology of $C^\bullet$ is well defined in $\mathcal{E}(\mathcal{A})$ (because $\mathcal{E}(\mathcal{A})$ is an Abelian category), and is explicitly given as follows:

$$\mathbb{H}^i(C^\bullet) = (d : C^{i-1} \to Z^i),$$

where Z^i denotes the Hilbertian $\mathcal{A}$-submodule of cocycles in C^i. It is called the i-th *extended L^2-cohomology* of the Hilbertian $\mathcal{A}$-complex $C^\bullet$. The projective part of the extended L^2-cohomology is the Hilbertian $\mathcal{A}$-module

$$P(\mathbb{H}^i(C^\bullet)) = Z^i/\overline{\operatorname{Im}(d : C^{i-1} \to C^i)} = H^i_{(2)}(C^\bullet)$$

which coincides with the reduced L^2-cohomology of the Hilbertian $\mathcal{A}$-complex $C^\bullet$. The torsion part of the extended L^2-cohomology is the morphism

$$T(\mathbb{H}^i(C^\bullet)) = (d : C^{i-1} \to \overline{\operatorname{Im}(d : C^{i-1} \to C^i)})$$

4.3. The extended twisted de Rham L^2-cohomology. Let M be a compact manifold and β a closed 1-form on M. Let $\mathcal{E} \to M$ be a flat Hilbertian $\mathcal{A}$-bundle over M. Following [Sh2] let us define the *extended twisted de Rham L^2-cohomology* of the complex

$$\left(\Omega^\bullet_{(2)}(M, \mathcal{E}), \nabla_\beta \right)$$

as the extended cohomology of any finitely generated Hilbertian $\mathcal{A}$-complex which is bounded chain homotopy equivalent to the given de Rham complex $\left(\Omega^\bullet_{(2)}(M, \mathcal{E}), \nabla_\beta \right)$. Since bounded chain homotopy is an equivalence relation, this is well defined. For example, let $E^j_\lambda(\beta)$ be the spectral projection of the operator $\Delta_{\beta,j}$. Let us fix $\varepsilon > 0$ and define $L^j = E^j_\varepsilon(\beta)$. Then L^j can be shown to be a finitely generated Hilbertian $\mathcal{A}$-module (cf. [Sh2]), and

$$\nabla_\beta : L^j \to L^{j+1}$$

is the bounded coboundary operator commuting with the $\mathcal{A}$-action. In particular, $\nabla^2_\beta = 0$. The Hilbertian $\mathcal{A}$-complex $(L^\bullet, \nabla_\beta)$ was earlier observed to be bounded chain homotopy equivalent to $\left(\Omega^\bullet_{(2)}(M, \mathcal{E}), \nabla_\beta \right)$. By definition, the *extended twisted de Rham L^2-cohomology* is the extended cohomology of the finitely generated Hilbertian $\mathcal{A}$-complex $(L^\bullet, \nabla_\beta)$. It is denoted by $\mathbb{H}^\bullet(M, \mathcal{E}_\beta)$ and is represented in $\mathcal{E}(\mathcal{A})$ by the morphism

$$\mathbb{H}^i(M, \mathcal{E}_\beta) = (\nabla_\beta : L^{i-1} \to Z^i)$$

where Z^i denotes the Hilbertian $\mathcal{A}$-submodule of cocycles in L^i. The projective part of the extended twisted de Rham L^2-cohomology is the Hilbertian $\mathcal{A}$-module

$$P(\mathbb{H}^i(M, \mathcal{E}_\beta)) = Z^i / \overline{\operatorname{Im}(\nabla_\beta : L^{i-1} \to L^i)} = H^i_{(2)}(L^\bullet, \nabla_\beta)$$

which coincides with the reduced de Rham L^2-cohomology of the Hilbertian $\mathcal{A}$-complex $(L^\bullet, \nabla_\beta)$. But as observed earlier, the reduced L^2-cohomology $H^i_{(2)}(L^\bullet, \nabla_\beta)$ coincides with the reduced twisted de Rham L^2-cohomology $H^i_{(2)}(M, \mathcal{E}_\beta)$. The torsion part of the extended twisted de Rham L^2-cohomology is the morphism

$$T(\mathbb{H}^i(M, \mathcal{E}_\beta)) = (\nabla_\beta : L^{i-1} \to \overline{\operatorname{Im}(\nabla_\beta : L^{i-1} \to L^i)}).$$

Alternatively we can use Witten deformation and take $L^j_W = \tilde{E}^j_\varepsilon(\beta + sdf)$ instead of L^j. Here f is a Morse function on M, $\tilde{E}^j_\varepsilon(\beta + sdf)$ is the spectral projection of $\frac{1}{s}\Delta_{\beta+sdf,j}$, $\varepsilon > 0$ is fixed, $s \gg 0$. Then again L^j_W will be a finitely generated Hilbertian $\mathcal{A}$-module ([Sh2]), and

$$\nabla_{\beta+sdf} : L^j_W \to L^{j+1}_W$$

is a bounded coboundary operator, defining a Hilbertian $\mathcal{A}$-complex $(L^\bullet_W, \nabla_{\beta+sdf})$ which is bounded chain homotopy equivalent to $(\Omega^\bullet_{(2)}, \nabla_\beta)$. The advantage of this approximation is that ε will be in the gap of the spectrum of the Laplacian $\frac{1}{s}\Delta_{\beta+sdf}$ for $s \gg 0$, so the spectral subspace L^j_W has better analytic and geometric properties.

4.4. **Minimal number of generators.** M. Farber [F] introduced the minimal number of generators as an invariant of virtual Hilbertian $\mathcal{A}$-modules. It is non-trivial on all non-trivial torsion modules and takes values in the nonnegative integers. More precisely, let χ be a virtually Hilbertian $\mathcal{A}$-module, and $\mu(\chi)$ be the smallest integer μ such that there exists an epimorphism from the direct sum of μ copies of $\ell^2(\mathcal{A})$ onto χ. Then $\mu(\chi)$ is called the *minimal number of generators* of χ and it has the following properties:

(1) $\mu(\chi) = 0$ if and only if $\chi = 0$; besides, $\mu(\chi) = \mu(P(\chi))$ if and only if $T(\chi) = 0$.
(2) If χ is projective, then $\mu(\chi) \geq \dim_\tau \chi$.
(3) if χ' is another virtually Hilbertian $\mathcal{A}$-module, then

$$\max\{\mu(\chi), \mu(\chi')\} \leq \mu(\chi \oplus \chi') \leq \mu(\chi) + \mu(\chi').$$

Some calculations of this invariant are done in [F], section 7.

The minimal number of generators seems to be useful for torsion objects provided $\mathcal{A}$ has a non-trivial center. It is easy to see that if $\mathcal{A}$ is a factor, then $\mu(\chi) = 1$ for any non-trivial torsion object.

Indeed, if $\mathcal{A}$ is a factor, then any torsion module χ can be represented by a morphism $\alpha : M' \to M$ where $\dim_\mathcal{A} M$ is arbitrarily small. (This follows from Farber's excision property [F] if we apply a spectral cut to α, removing a part of the spectrum of $|\alpha|$ outside $[0, \epsilon]$ for small ϵ.) It follows that there is an epimorphism from M (identified with $0 \to M$) onto χ. Now by the fundamental property of Hilbert modules over factors (see e.g. [Di] or [T]) we can embed M into $l^2(\mathcal{A})$ as a Hilbert $\mathcal{A}$-module, and, therefore, produce epimorphism (by orthogonal projection) of $l^2(\mathcal{A})$ onto M, hence onto χ. This means that $\mu(\chi) \leq 1$.

The following abstract theorem is due to Farber [F], theorem 8.1. It has been rephrased here in terms of cohomology.

Theorem 4.1. *Let*

$$C^\bullet : \quad \cdots \to C^{i-1} \xrightarrow{d} C^i \xrightarrow{d} C^{i+1} \to \cdots$$

be a free finitely generated Hilbertian chain complex in $\mathcal{E}(\mathcal{A})$. Then for any integer i the following inequality holds:

$$\dim_\tau(C^i) \geq \mu[\mathbb{H}^i(C) \oplus T(\mathbb{H}^{i+1}(C))].$$

The first part of the following Theorem was essentially proved by Farber [F] using combinatorial methods. Our method is different (we use analysis), and also the second part of the Theorem is new.

Theorem 4.2. *Let M be a compact manifold, f a Morse function on M and θ a Morse 1-form on M. Let $\mathcal{E} \to M$ be a flat Hilbertian $\mathcal{A}$-bundle over M such that its fiber E is a free finitely generated Hilbertian $\mathcal{A}$-module.*

(i) *(L^2 Morse-Farber Inequalities).*

$$m_j(f) \geq (\dim_\tau E)^{-1} \cdot \mu[\mathbb{H}^j(M, \mathcal{E}_\theta) \oplus T(\mathbb{H}^{j+1}(M, \mathcal{E}_\theta))]$$

for $j = 0, 1, 2, \ldots$.

(ii) *(Asymptotic L^2 Morse-Farber Inequalities). For $s \gg 0$, one has*

$$m_j(\theta) \geq (\dim_\tau E)^{-1} \cdot \mu[\mathbb{H}^j(M, \mathcal{E}_{s\theta}) \oplus T(\mathbb{H}^{j+1}(M, \mathcal{E}_{s\theta}))]$$

for $j = 0, 1, 2, \ldots$.

Proof. Let α, β be closed 1-forms on M, and besides α is a Morse form. It is convenient to use the Witten "small eigenvalues" approximation $(L_W^\bullet, \nabla_{\beta+s\alpha})$, $s \gg 0$, of the complex $(\Omega_{(2)}^\bullet(M, \mathcal{E}_\beta), \nabla_\beta)$ as described at the end of sect. 4.3.

Assume first that $\mathcal{A}$ is a factor. Since E is a free Hilbertian $\mathcal{A}$ module, it follows that L_W^j is a free Hilbertian $\mathcal{A}$-module for all j, because $\dim_\tau L_W^j$ is an integer by Lemma 3.10 and the discussion before it. For general $\mathcal{A}$ we still can prove that L_W^j will be free by taking the factor decomposition of E and using arguments from [Sh2].

An alternative way to prove that L_W^j is free: observe that a small refinement of the arguments in [Sh] shows that L_W^j is isomorphic (as a Hilbertian $\mathcal{A}$-module) to Ker $K_{(j)}$, where $K_{(j)}$ is the model operator corresponding to the Hamiltonian $H = \frac{1}{s}\Delta_{\beta+s\alpha,j}$ as described in section 3.2.

The theorem follows if we apply Theorem 4.1 to the the free finitely generated Hilbertian $\mathcal{A}$-complex $(L_W^\bullet, \nabla_{\beta+s\alpha})$, for $s \gg 0$, as in Proposition 3.11. Setting $\alpha = df$ and $\beta = \theta$, we observe that $[\beta + s\alpha] = [\theta]$ for all s, proving part (i).

Setting $\alpha = \theta$ and $\beta = 0$, we observe that $[\beta + s\alpha] = s[\alpha]$, proving part (ii). $\square$

The following corollary can be viewed as a quantitative version of Proposition 3.13.

Corollary 4.3. *Let M be a compact manifold, f a Morse function on M and θ a closed 1-form on M. Let $\mathcal{E} \to M$ be a flat Hilbertian $\mathcal{A}$-bundle over M such that its fiber is a free finitely generated Hilbertian $\mathcal{A}$-module E. Let $\lambda_0(\Delta_{\theta,j})$ denote the bottom of the L^2-spectrum of the twisted Laplacian $\Delta_{\theta,j}$ acting on the complement of its L^2-kernel.*

(a) *Suppose that $\lambda_0(\Delta_{\theta,j}) = 0$, i.e. there is no spectral gap at zero. Then*

$$m_j(f) \; > \; (\dim_\tau E)^{-1} \cdot \mu[P(\mathbb{H}^j(M, \mathcal{E}_\theta))] \geq (\dim_\tau E)^{-1} \cdot b^j_{(2)}(M, \mathcal{E}_\theta).$$

for every Morse function f on M.

(b) *Suppose that θ is a Morse 1-form on M and $\lambda_0(\Delta_{s\theta,j}) = 0$ for $s \gg 0$. Then*

$$m_j(\theta) \; > \; (\dim_\tau E)^{-1} \cdot \mu[P(\mathbb{H}^j(M, \mathcal{E}_{s\theta}))] \geq (\dim_\tau E)^{-1} \cdot b^j_{(2)}(M, \mathcal{E}_{s\theta}).$$

for $s \gg 0$.

Proof. Since $\lambda_0(\Delta_{\theta,j}) = 0$, it follows that $T(\mathbb{H}^j(M, \mathcal{E}_\theta))$ is a non-trivial virtual Hilbertian $\mathcal{A}$-module. By property (1) of the minimal number of generators, one has $\mu(\mathbb{H}^j(M, \mathcal{E}_\theta)) > \mu(P(\mathbb{H}^j(M, \mathcal{E}_\theta)))$. By Theorem 4.2, part (i), and property (3) of the minimal number of generators, one has

$$m_j(f) \; \geq \; (\dim_\tau E)^{-1} \cdot \mu[\mathbb{H}^j(M, \mathcal{E}_\theta)] > (\dim_\tau E)^{-1} \cdot \mu[P(\mathbb{H}^j(M, \mathcal{E}_\theta))].$$

The last inequality in part (a) follows from property (2) of the minimal number of generators.

Part (b) is proved similarly. $\qquad\square$

5. CALCULATIONS.

In this section, we do some calculations in the special case of the Hilbertian $(\mathcal{U}(\pi) - \pi)$-bimodule $\ell^2(\pi)$. Let $\mathcal{E} \to M$ denote the associated flat Hilbertian $\mathcal{U}(\pi)$-bundle over M. Then it is well known that the Hilbertian $\mathcal{U}(\pi)$-complexes $\left(\Omega^\bullet_{(2)}(M, \mathcal{E}), \nabla\right)$ and $\left(\Omega^\bullet_{(2)}(\widetilde{M}), d\right)$ are canonically isomorphic, where $p : \widetilde{M} \to M$ denotes the universal cover of M. This isomorphism also establishes that if θ is a closed 1-form on M, then $\left(\Omega^\bullet_{(2)}(M, \mathcal{E}), \nabla_\theta\right)$ and $\left(\Omega^\bullet_{(2)}(\widetilde{M}), d_\theta\right)$ are canonically isomorphic, where $d_\theta = d + e(p^*\theta)$. Here $e(p^*\theta)$ denotes exterior multiplication by the closed 1-form $p^*\theta$. In this case, we shall denote the twisted L^2-invariants by $R(\widetilde{M}, \theta) = R(M, \mathcal{E}_\theta)$, where R denotes any particular L^2-invariant as in the previous sections.

We begin with the following basic lemma.

Lemma 5.1. *Let M be a closed connected manifold and $\pi_1(M)$ be infinite. Then*

$$H^0_{(2)}(\widetilde{M}, \theta) = \{0\} = H^n_{(2)}(\widetilde{M}, \theta),$$

where $n = \dim_\mathbb{R} M$.

Proof. Since $H^0_{(2)}(\widetilde{M}, \theta) = \mathrm{Ker}\, \Delta_{\theta,0} = \mathrm{Ker}\, d_\theta$ on $\Omega^0_{(2)}(\widetilde{M})$, we will first prove that $\mathrm{Ker}\, d_\theta = \{0\}$ on L^2-functions on $\widetilde{M}$. Let $g \in \mathrm{Ker}\, d_\theta$. Let $p^*\theta = df$, $f \in C^\infty(\widetilde{M})$. Since

$$0 = d_\theta g = e^{-f} d(e^f g),$$

we see that $d(e^f g) = 0$, hence $e^f g = c = \mathrm{const}$ and also $g \in L^2(\widetilde{M})$.

Now let us check that the inclusion $g = ce^{-f} \in L^2(\widetilde{M})$ is possible only if $c = 0$. Let F be a fundamental domain of the action of $\pi = \pi_1(M)$ on $\widetilde{M}$ by deck transformations, so that

$$\widetilde{M} = \cup\{\gamma F| \; \gamma \in \pi\}$$

up to a set of measure 0. We can assume F to be open and connected. Let us fix a point $P \in F$ and assume that $f(P) = 0$. Then $f(\gamma P) = \int_{l_\gamma} \theta$ where l_γ is a loop representing γ in $\pi_1(M)$ with the base point P. Clearly $f(\gamma_1 \gamma_2 P) = f(\gamma_1 P) + f(\gamma_2 P)$. Denote $\phi(\gamma) = \exp(f(\gamma P))$. Then $\phi(\gamma_1 \gamma_2) = \phi(\gamma_1)\phi(\gamma_2)$, so $\phi : \pi \to (0, \infty)$ is a group homomorphism.

It is easy to see that for any $Q \in F$

$$f(\gamma Q) - f(Q) = \int_{l_\gamma} \theta = f(\gamma P) - f(P) = f(\gamma P),$$

hence

$$\exp(-f(\gamma Q)) = \phi(\gamma^{-1})\exp(-f(Q)), \quad Q \in F, \ \gamma \in \pi .$$

Integrating over F and summing over all $\gamma \in \pi$, we see that

$$\int_{\widetilde{M}} |\exp(-f(x))|^2 dx = \left(\sum_{\gamma \in \pi} |\phi(\gamma)|^2 \right) \int_F |\exp(-f(x))|^2 dx ,$$

where dx denotes any π-invariant smooth measure with a positive density on $\widetilde{M}$. It follows that the inclusion $e^{-f} \in L^2(\widetilde{M})$ is equivalent to $\sum_{\gamma \in \pi} |\phi(\gamma)|^2 < \infty$ which is obviously impossible for infinite π. Therefore the constant c above should vanish and $g \equiv 0$.

We conclude that $H^0_{(2)}(\widetilde{M}, \theta) = \{0\}$.

By Poincaré duality argument (applied on $\widetilde{M}$ which is orientable!) we obtain

$$H^n_{(2)}(\widetilde{M}, \theta) = H^0_{(2)}(\widetilde{M}, -\theta) = \{0\} .$$

$\square$

Our next result proves the upper semi-continuity property of the twisted L^2 Betti numbers as a function on $H^1(M, \mathbb{R})$.

Lemma 5.2. *The function $H^1(M, \mathbb{R}) \to \mathbb{R}$ given by $[\theta] \mapsto b^j_{(2)}(\widetilde{M}, \theta)$ is upper semi-continuous.*

Proof. We identify $H^1(M, \mathbb{R}) \cong \mathcal{H}^1(M, \mathbb{R})$. The proof is based on the fact that $\theta \mapsto \Delta_{\theta,j}$ is an analytic family. Recall that a function $f : \mathcal{H}^1(M, \mathbb{R}) \to \mathbb{R}$ is said to be upper semi-continuous at $\alpha \in \mathcal{H}^1(M, \mathbb{R})$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any θ satisfying $|\theta - \alpha| < \delta$, we have

$$f(\theta) < f(\alpha) + \varepsilon$$

For each $\alpha \in \mathcal{H}^1(M, \mathbb{R})$, the operator $\Delta_{\alpha,j}$ is π-Fredholm in $\Omega^j_{(2)}(\widetilde{M})$. (Here $\pi = \pi_1(M)$.) This means firstly that $\dim_\pi \operatorname{Ker} \Delta_{\alpha,j} < \infty$ where $\dim_\pi = \dim_\tau$ for the canonical trace τ on $\mathcal{U}(\pi)$, and secondly that for any $\varepsilon > 0$ there exists a closed π-invariant subspace $L_\varepsilon \subset \operatorname{Im} \Delta_{\alpha,j}$ such that $\dim_\pi(\overline{\operatorname{Im} \Delta_{\alpha,j}} \ominus L_\varepsilon) < \varepsilon$ i.e. $\overline{\operatorname{Im} \Delta_{\alpha,j}} = L_\varepsilon \oplus L'_\varepsilon$ with $\dim_\pi L'_\varepsilon < \varepsilon$.

Note also that

$$\Omega^j_{(2)}(\widetilde{M}) = \operatorname{Ker} \Delta_{\alpha,j} \oplus \overline{\operatorname{Im} \Delta_{\alpha,j}} .$$

By the closed graph theorem, there exists $C > 0$ such that

$$\|\Delta_{\alpha,j} u\| \geq C\|u\|, \quad u \in L_\varepsilon .$$

It follows that for any $\delta' > 0$ we can choose $\delta > 0$ such that

$$\|\Delta_{\theta,j}u\| \geq (C - \delta')\|u\| \quad \text{whenever} \quad u \in L_\varepsilon \quad \text{and} \quad |\theta - \alpha| < \delta.$$

Therefore

$$\dim_\pi(\text{Ker } \Delta_{\theta,j}) \leq \text{codim}_\pi(L_\varepsilon) < \dim_\pi(\text{Ker } \Delta_{\alpha,j}) + \varepsilon.$$

$\square$

Let $\mathcal{L}_\mathbb{Q}$ denote the class of finitely presented discrete groups π such that the von Neumann dimension of the kernel of any π-invariant operator acting on $\ell^2(\pi) \otimes \mathbb{C}^l$ and which comes from the group algebra $\mathbb{C}(\pi)$, is a rational number. $\mathcal{L}_\mathbb{Q}$ contains all elementary amenable groups and extensions of groups in $\mathcal{L}_\mathbb{Q}$ by right orderable groups. Let $\mathcal{L}_\mathbb{Z} \subset \mathcal{L}_\mathbb{Q}$ denote the class of finitely presented discrete groups π such that the von Neumann dimension of the kernel of any π-invariant operator acting on $\ell^2(\pi) \otimes \mathbb{C}^l$ and which comes from the group algebra $\mathbb{C}(\pi)$, is an integer. It is known that $\mathcal{L}_\mathbb{Z}$ contains all torsion-free elementary amenable groups and torsion-free extensions of groups in $\mathcal{L}_\mathbb{Z}$ by torsion-free right orderable groups. It has been conjectured that $\mathcal{L}_\mathbb{Q}$ contains all finitely presented groups and $\mathcal{L}_\mathbb{Z}$ contains all torsion-free finitely presented groups. (See Cohen [C], Donnelly [Don] and especially Linnell [Li] for further information on this discussion.)

Corollary 5.3. *Suppose that $b^j_{(2)}(\widetilde{M}) = 0$ and $\pi_1(M) \in \mathcal{L}_\mathbb{Z}$. Then there is a $\delta > 0$ such that $b^j_{(2)}(\widetilde{M}, [\theta]) = 0$ for all $[\theta]$ with $\|[\theta]\| < \delta$.*

5.1. **Flat manifolds.** Let θ denote the S^1-invariant nowhere zero 1-form on the circle $S^1 = \mathbb{R}/\mathbb{Z}$, which represents a generator for $H^1(S^1, \mathbb{R})$. Then any other element of $H^1(S^1, \mathbb{R})$ is represented by $s\theta$, $s \in \mathbb{R}$. We can assume that $|\theta(x)| = 1$ in the canonical metric for all $x \in S^1$. Choosing the sign, we can take $\theta = dx$ where x is the canonical coordinate in $\mathbb{R}$.

On $\mathbb{R}$, and on functions, one calculates for the case of the standard flat metric

$$\Delta_{s\theta,0} = -\frac{\partial^2}{\partial x^2} + s(\mathcal{L}_V + \mathcal{L}_V^*) + s^2\,,$$

where $V = \frac{\partial}{\partial x}$ is dual to dx. So $\mathcal{L}_V = \frac{\partial}{\partial x} = -\mathcal{L}_V^*$. That is,

$$\Delta_{s\theta,0} = -\frac{\partial^2}{\partial x^2} + s^2\,.$$

We get a spectral resolution of $\Delta_{s\theta,0}$ using the Fourier transform, and one calculates that

$$\sigma(\Delta_{s\theta,0}) = [s^2, \infty) = \sigma(\Delta_{s\theta,1}).$$

We conclude that

$$\lambda_{0,0}(\Delta_{s\theta}) \quad = \quad \lambda_{0,1}(\Delta_{s\theta}) \quad = \quad s^2$$

$$\alpha_0(\widetilde{S}^1, s[\theta]) \quad = \quad \alpha_1(\widetilde{S}^1, s[\theta]) \quad = \quad \begin{cases} \frac{1}{2} & \text{if} \quad s = 0 \\ \infty & \text{if} \quad s \neq 0 \end{cases}$$

$$b^0_{(2)}(\widetilde{S}^1, s[\theta]) \quad = \quad b^1_{(2)}(\widetilde{S}^1, s[\theta]) \quad = \quad 0 \quad \text{for all} \quad s.$$

Here $\lambda_{0,j}(A)$ denotes the bottom of the spectrum of the operator A acting on L^2-forms of degree j. Generalizing this calculation to the higher dimensions, one has

Proposition 5.4. . *Let θ be a closed 1-form on a compact flat manifold M of dimension n (i.e. M is a flat n-dimensional torus). Then*

(1) $b_{(2)}^j(\widetilde{M}, [\theta]) = 0$ *for $j = 0, \ldots, n$.*

(2) $\alpha_j(\widetilde{M}, [\theta]) = \begin{cases} \frac{n}{2} & if \quad [\theta] = 0, \\ \infty & if \quad [\theta] \neq 0. \end{cases}$

(3) $\sigma(\Delta_{\theta,j}) = [\|\theta\|^2, \infty)$.

(4) $\lambda_{0,j}(\Delta_\theta) = \|\theta\|^2$.

5.2. **Nil manifolds.** Let G be a connected and simply-connected nilpotent Lie group and Γ be a torsion-free, discrete, cocompact subgroup. Then the space of differential forms on a nilmanifold $_\Gamma \backslash G$ is isomorphic to the space of Γ-invariant differential forms on G, that is,

$$\Omega^*(_\Gamma \backslash G) \cong \Omega^*(G)^\Gamma$$

Let $\mathcal{J}$ denote the Lie algebra of G. The Lie algebra cochains are

$$C^*(\mathcal{J}) = \Omega^*(G)^G \hookrightarrow \Omega^*(G)^\Gamma.$$

Nomizu [Nom] proved that the inclusion above induces an isomorphism in cohomology, that is,

$$H^*(\mathcal{J}) \cong H^*(_\Gamma \backslash G)$$

In other words elements of $H^*(_\Gamma \backslash G)$ are represented by nowhere zero, G-invariant differential forms on G. In particular, if $\alpha \in H^1(_\Gamma \backslash G)$ and α is not trivial, then α is represented by a nowhere zero closed G-invariant 1-form on G, which induces a nowhere zero closed 1-form θ on $\Gamma \backslash G$. The asymptotic Morse inequalities imply that

$$b_{(2)}^j(\widetilde{\Gamma \backslash G}, s[\theta]) = 0 \quad \text{for all} \quad j = 0, 1, \ldots, \dim G, \quad \text{and} \quad s \gg 0.$$

In fact, one sees in the proof of these inequalities that a spectral gap near zero develops for $\Delta_{s\theta,j}$, for all $s \gg 0$, since θ is nowhere zero. So

$$\alpha_j(\widetilde{\Gamma \backslash G}, s[\theta]) = \infty \quad \text{for} \quad j = 0, \ldots, \dim G, \quad \text{and} \quad s \gg 0.$$

Summarizing, one has

Proposition 5.5. . *Let θ be a closed 1-form on a compact Nil manifold M of dimension n. Then*

(1) $b_{(2)}^j(\widetilde{M}, s[\theta]) = 0$ *for $j = 0, \ldots, n$, and for all $s \gg 0$.*

(2) $\alpha_j(\widetilde{M}, s[\theta]) = \infty$ *for $j = 0, \ldots, n$, and for all $s \gg 0$.*

(3) $\lambda_{0,j}(\Delta_{s\theta}) > 0$ *for $j = 0, \ldots, n$, and for all $s \gg 0$.*

5.3. **Mapping cylinders.** Let $p : M \to S^1$ be a closed n-dimensional manifold which is a fiber bundle over the circle S^1. Since there is a nowhere zero closed 1-form θ_0 on S^1, we can consider $p^*(\theta_0)$ which is a nowhere zero closed 1-form on M. Arguments similar to that given for Nil manifolds yield

Proposition 5.6. *For a closed n-dimensional manifold $p : M \to S^1$ which is a fiber bundle over S^1, in the notations described above one has*

(1) $b^j_{(2)}(\widetilde{M}, s[p^*(\theta_0)]) = 0$ *for $j = 0, \dots, n$, and all $s \gg 0$.*

(2) $\alpha_j(\widetilde{M}, s[p^*(\theta_0)]) = \infty$ *for $j = 0, \dots, n$, and all $s \gg 0$.*

(3) $\lambda_{0,j}(\Delta_{s,p^*(\theta_0)}) > 0$ *for $j = 0, \dots, n$, and all $s \gg 0$.*

5.4. **2-manifolds.**

Proposition 5.7. *Let M be a compact 2-dimensional manifold. Then the twisted L^2 Betti functions $b^j_{(2)}(\widetilde{M}, [\theta])$ are constant on $H^1(M, \mathbb{R})$ for $j = 0, 1, 2$. More precisely, $b^1_{(2)}(\widetilde{M}, [\theta]) = -\chi(M)$ and $b^0_{(2)}(\widetilde{M}, [\theta]) = b^2_{(2)}(\widetilde{M}, [\theta]) = 0$.*

Proof. If genus$(M) = 0$, then $H^1(M, \mathbb{R}) = 0$ and the result is clear. If genus$(M) > 0$, then by Proposition 3.12 one has

$$b^0_{(2)}(\widetilde{M}, [\theta]) - b^1_{(2)}(\widetilde{M}, [\theta]) + b^2_{(2)}(\widetilde{M}, [\theta]) = \chi(M).$$

Since $b^0_{(2)}(\widetilde{M}, [\theta]) = b^2_{(2)}(\widetilde{M}, [\theta]) = 0$ by Lemma 5.1, one deduces the proposition. $\square$

An immediate corollary is

Corollary 5.8. *Let θ be a Morse 1-form on a compact 2-dimensional manifold M. Then $m_1(\theta) \geq -\chi(M)$.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ADELAIDE, ADELAIDE 5005, AUSTRALIA
E-mail address: `vmathai@maths.adelaide.edu.au`

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MASS., USA
E-mail address: `shubin@neu.edu`